Prime Mystery

The Life and Mathematics of Sophie Germain

Dora E. Musiełak, Ph.D.

… Two hundred years ago, **Sophie Germain** won a **Prize of Mathematics** for her mathematical theory of vibrating elastic surfaces …

Years earlier, she had begun innovative analysis to prove **Fermat's Last Theorem** …

… Sophie Germain had no formal education …

What did she do to achieve so much, and how?

What mathematics did she advance, and why?

Read ***Prime Mystery*** and discover Sophie Germain's fascinating and unconventional life, and how she contributed to both applied and pure mathematics

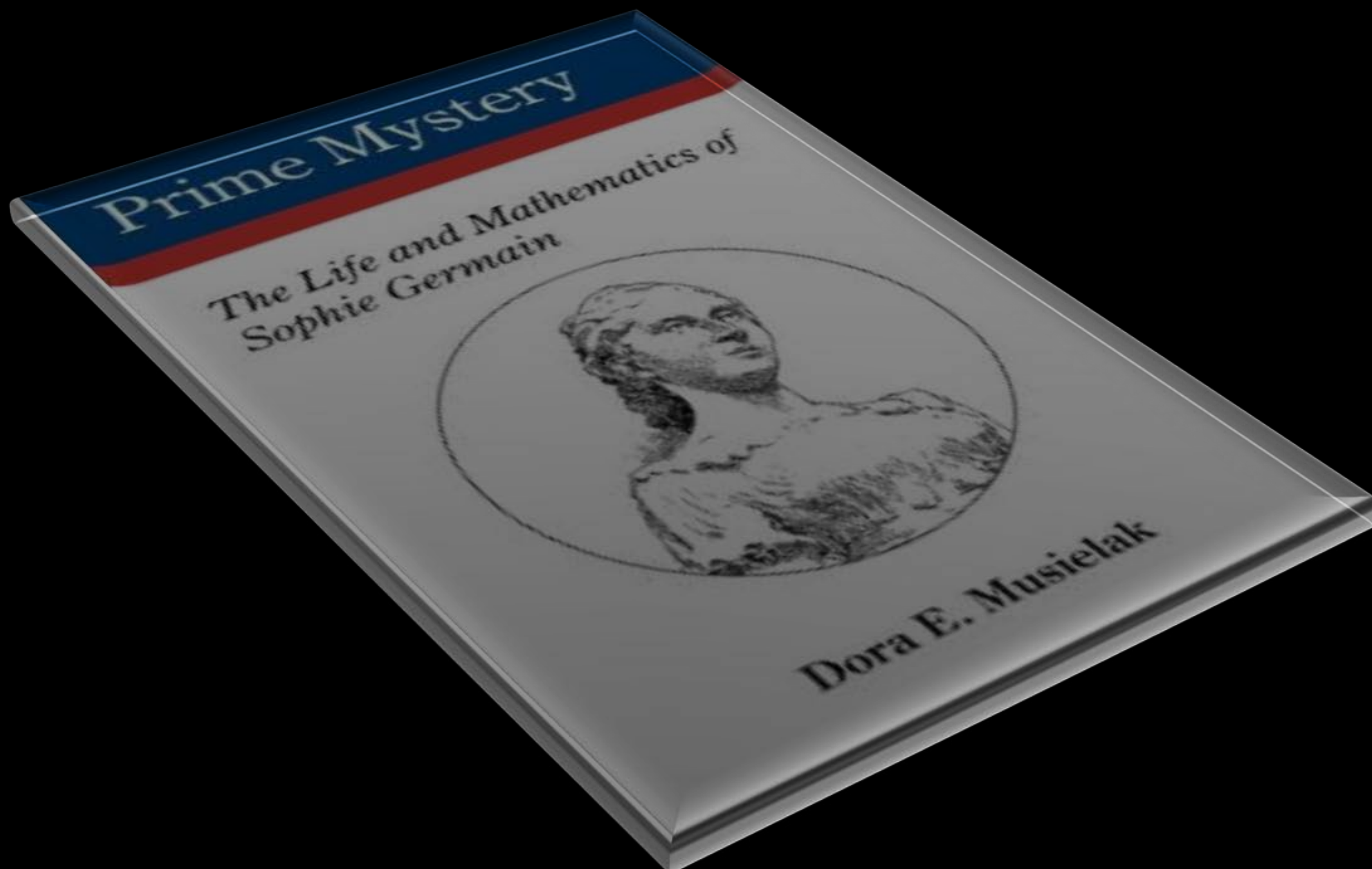


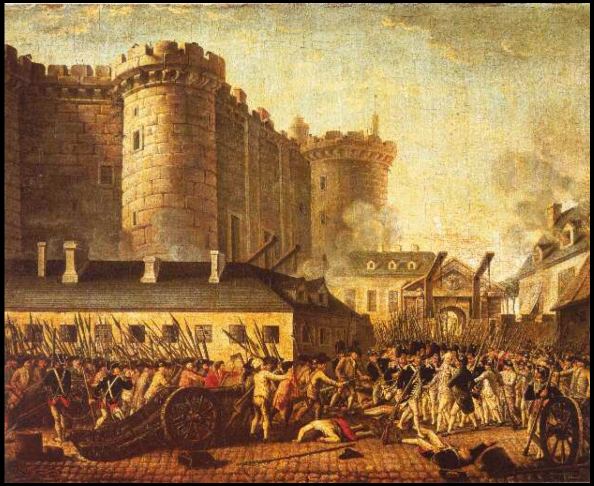
TABLE OF CONTENT

Unforgettable Childhood	3
Lessons from l'École Polytechnique	5
Chladni and Acoustics	7
Euler and the Bernoullis	11
Germain and Her Biharmonic Equation	13
Experiments with Vibrating Plates	17
Elasticity Theory After Germain	19
Germain and Fermat's Last Theorem	23
Pensées de Germain	29
Friends, Rivals, and Mentors	31
The Last Years	37
Unanswered Questions	41
Princess of Mathematics	43


UNFORGETTABLE CHILDHOOD
Paris 1776-1789
Reign of Louis XVI and Marie Antoinette

French Revolution and Reign of Terror

1789 - 1794



Sophie Germain came of age during the most brutal years of the revolution.

Chapter 3 focuses on her self-studies, giving details of mathematicians of that era. It also highlights how the Institute of France was founded amidst civil chaos.

How did Sophie Germain learn mathematics?

5
Lessons from l'École Polytechnique

Les leçons suivantes offrent un cours d'analyse sur cette partie du calcul qu'on nomme communément infinitésimale ou transcendante, et qui n'est proprement que le calcul des fonctions. Lagrange, 1804

Following the brutal Reign of Terror, an emergency council was set up in Paris. Its main task was the creation of a new engineering school called the École centrale des Travaux publics, which had the objective to train engineers, both civilian and military. Four hundred students quickly enrolled, with "revolutionary courses" in mathematics and chemistry as the foundation of their studies.[20] The school opened its doors on 21 December 1794, a Sunday.[21] The original building was the Hôtel de Lassay, a stately mansion overlooking the Seine River, right next to the Palais Bourbon (now the National Assembly). The luxurious Hôtel de Lassay,[22] which had been confiscated as national property in 1792, housed the new engineering school from 1794 to 1804. In September 1795, by a decree of the Convention the school name changed to École Polytechnique, presumably intended to convey the idea of a plurality of techniques.[23]

[20] Grattan-Guinness, I., "The Ecole Polytechnique, 1794-1850: Differences over Educational Purpose and Teaching Practice." The American Mathematical Monthly, Vol. 112, No. 3 (Mar., 2005), pp. 233-250.
[21] 1er nivôse an III (Dimanche, 21 décembre 1794), Ouverture des cours de l'École Centrale des Travaux publics.
[22] It is now residence of the president of the National Assembly. The building located on rue de l'Université faces the Jardin des Tuileries to the east and the Champs-Élysées on the west, in 7th arrondissement.
[23] Grattan-Guinness, "The Ecole," p. 233.

contributions from Lagrange.[29] Lagrange added approximately one hundred pages of work on Diophantine equations. This book appeared in Paris in 1794, when Sophie Germain was eighteen years old. I'd like to think that she studied this work and thus was prepared for Lagrange's analysis. In fact, the students applying for entrance to the École were required to know arithmetic and algebra, including the resolution of polynomial equations of up to the fourth degree; geometry, including trigonometry, the application of algebra to geometry, and conic sections.

Lagrange's Lecture Notes 1797 to 1799

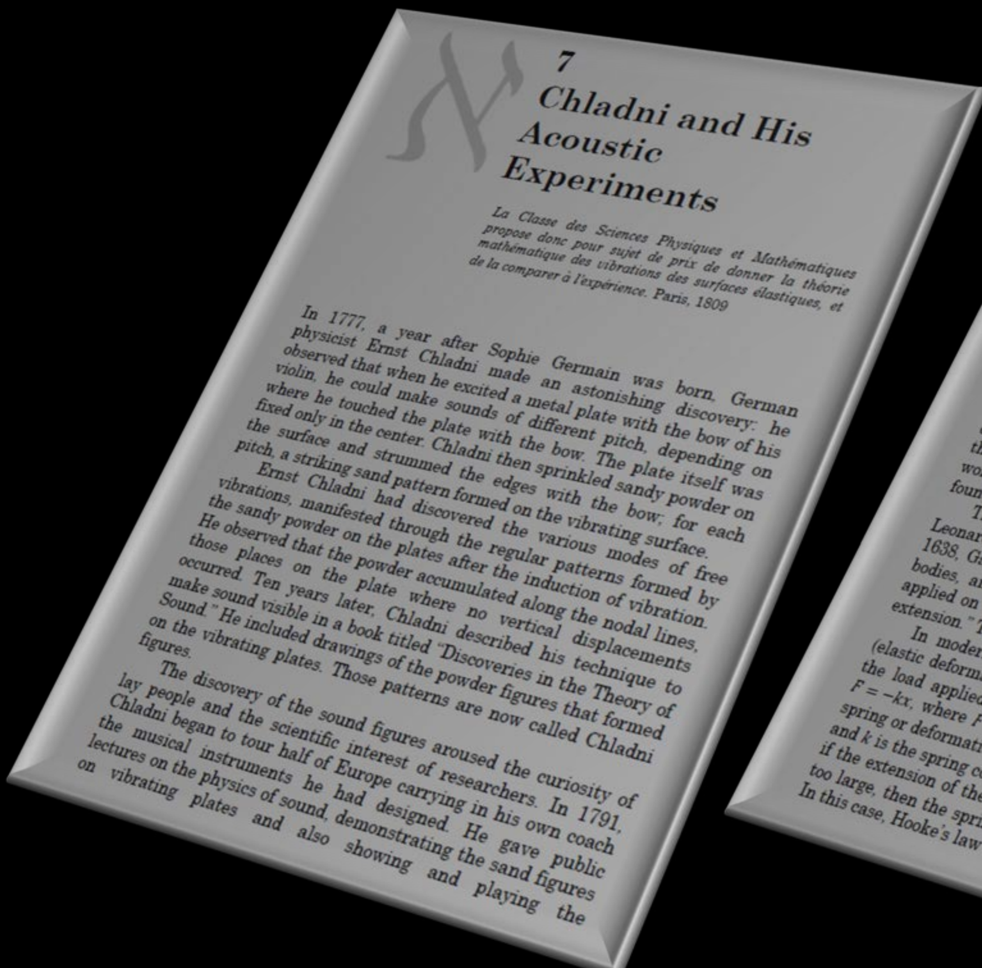
The first issue of the Journal de l'École Polytechnique is dated 1794 but was published in the spring of 1795 (mois de Germinal, an III). It begins with a lecture by Monge on *stéréotomie (technique traditionnelle de la coupe des matériaux de construction)*, and it contains a lecture by Gaspard de Prony dealing with analysis applied to mechanics (*Cours d'analyse appliqué à la mécanique*). Mastering the material in these cahiers was intended to prepare students for a degree in engineering.

The second issue of 1795 (mois de Floréal et de Prairial, Nivôse, an IV) contains an announcement by de Prony about a basic course of analysis by Lagrange (*Notice sur un cours élémentaire d'analyse fait par Lagrange, par R. Prony*). The third issue (*Messidor, Thermidor et Fructidor, an III*) contains the organization chart of the École, providing the names of all instructors and the subject matter included in the program of study. It begins with lessons on analysis by Gaspard de Prony.

The class notes of Lagrange's first lectures appeared in the fifth cahier (dated 1797) published in the summer of 1798

[29] Euler, L., *Élémens d'algèbre, par Léonard Euler, traduits de l'allemand avec des notes et des additions*, Contributeurs : Bernoulli, Jean (1744-1807). Traducteur ; Lagrange, Joseph-Louis (1736-1813). Traducteur. Éditeur : Bruyset aîné (Lyon), 1794.

What inspired Sophie Germain to compete in the prize of mathematics that she won?



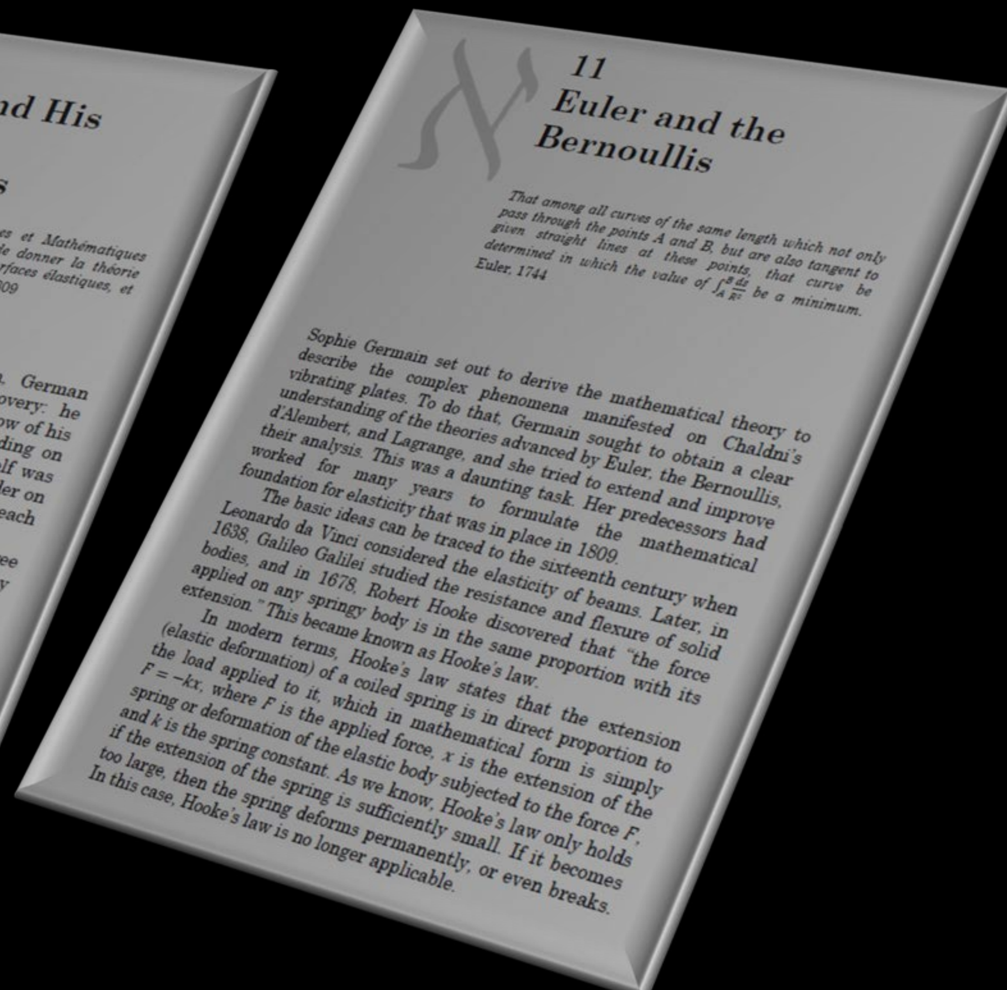
7
Chladni and His Acoustic Experiments

La Classe des Sciences Physiques et Mathématiques propose donc pour sujet de prix de donner la théorie mathématique des vibrations des surfaces élastiques, et de la comparer à l'expérience. Paris, 1809

In 1777, a year after Sophie Germain was born, German physicist Ernst Chladni made an astonishing discovery: he observed that when he excited a metal plate with the bow of his violin, he could make sounds of different pitch, depending on where he touched the plate with the bow. The plate itself was fixed only in the center. Chladni then sprinkled sandy powder on the surface and strummed the edges with the bow; for each pitch, a striking sand pattern formed on the vibrating surface.

Ernst Chladni had discovered the various modes of free vibrations, manifested through the regular patterns formed by the sandy powder on the plates after the induction of vibration. He observed that the powder accumulated along the nodal lines, those places on the plate where no vertical displacements occurred. Ten years later, Chladni described his technique to make sound visible in a book titled "Discoveries in the Theory of Sound." He included drawings of the powder figures that formed on the vibrating plates. Those patterns are now called Chladni figures.

The discovery of the sound figures aroused the curiosity of lay people and the scientific interest of researchers. In 1791, Chladni began to tour half of Europe carrying in his own coach the musical instruments he had designed. He gave public lectures on the physics of sound, demonstrating the sand figures on vibrating plates and also showing and playing the



11
Euler and the Bernoullis

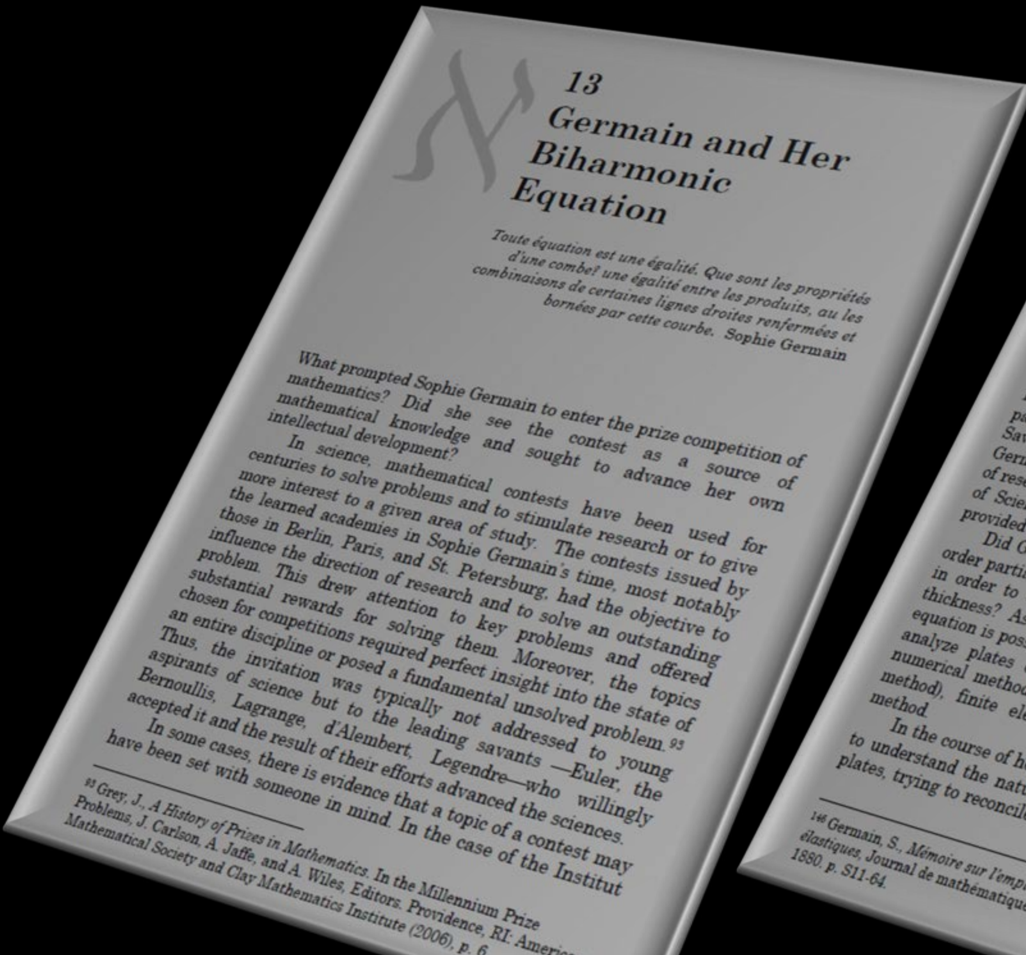
That among all curves of the same length which not only pass through the points A and B, but are also tangent to given straight lines at these points, that curve be determined in which the value of $\int_A^B \frac{ds}{R^2}$ be a minimum. Euler, 1744

Sophie Germain set out to derive the mathematical theory to describe the complex phenomena manifested on Chaldni's vibrating plates. To do that, Germain sought to obtain a clear understanding of the theories advanced by Euler, the Bernoullis, d'Alembert, and Lagrange, and she tried to extend and improve their analysis. This was a daunting task. Her predecessors had worked for many years to formulate the mathematical foundation for elasticity that was in place in 1809.

The basic ideas can be traced to the sixteenth century when Leonardo da Vinci considered the elasticity of beams. Later, in 1638, Galileo Galilei studied the resistance and flexure of solid bodies, and in 1678, Robert Hooke discovered that "the force applied on any springy body is in the same proportion with its extension." This became known as Hooke's law.

In modern terms, Hooke's law states that the extension (elastic deformation) of a coiled spring is in direct proportion to the load applied to it, which in mathematical form is simply $F = -kx$, where F is the applied force, x is the extension of the spring or deformation of the elastic body subjected to the force F, and k is the spring constant. As we know, Hooke's law only holds if the extension of the spring is sufficiently small. If it becomes too large, then the spring deforms permanently, or even breaks. In this case, Hooke's law is no longer applicable.

What did Sophie Germain do to develop her mathematical theory and win the prize of mathematics?



13

Germain and Her Biharmonic Equation

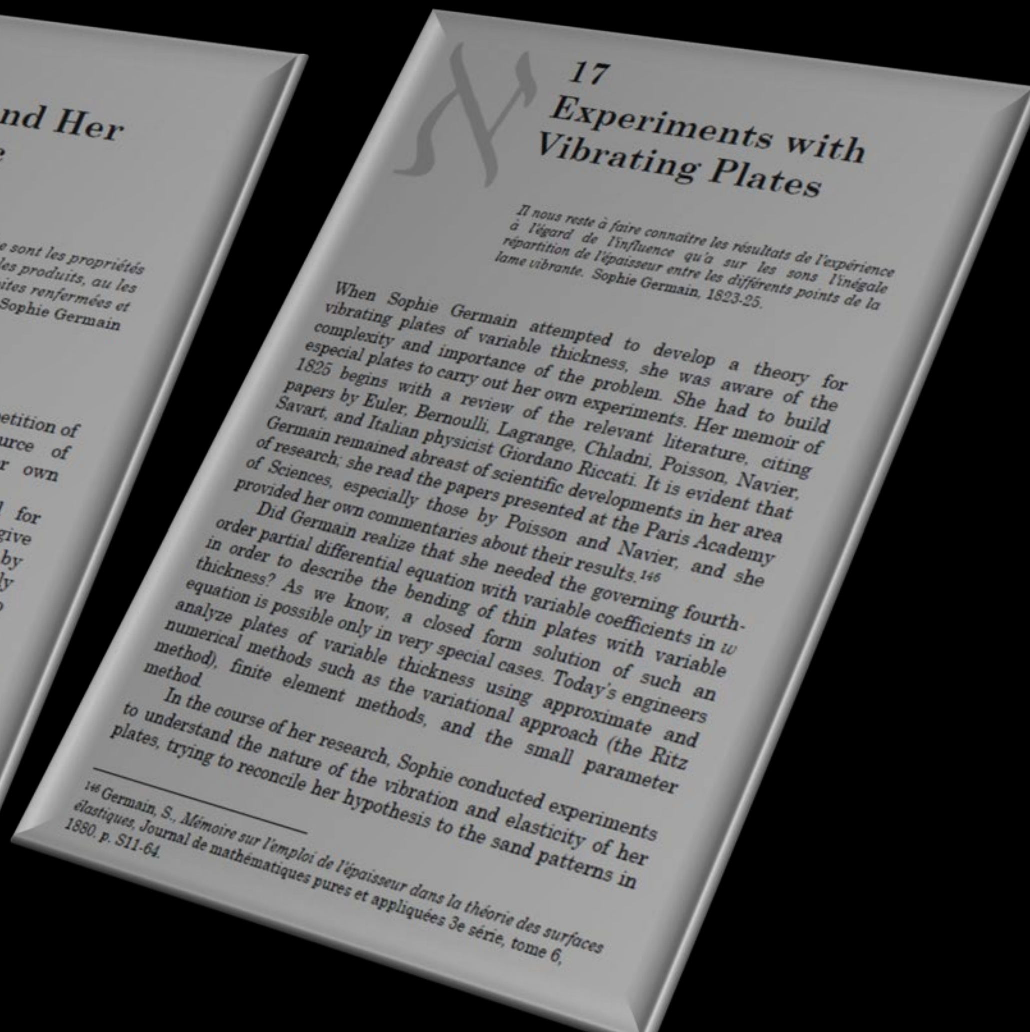
Toute équation est une égalité. Que sont les propriétés d'une courbe? une égalité entre les produits, ou les combinaisons de certaines lignes droites renfermées et bornées par cette courbe. Sophie Germain

What prompted Sophie Germain to enter the prize competition of mathematics? Did she see the contest as a source of mathematical knowledge and sought to advance her own intellectual development?

In science, mathematical contests have been used for centuries to solve problems and to stimulate research or to give more interest to a given area of study. The contests issued by the learned academies in Sophie Germain's time, most notably those in Berlin, Paris, and St. Petersburg, had the objective to influence the direction of research and to solve an outstanding problem. This drew attention to key problems and offered substantial rewards for solving them. Moreover, the topics chosen for competitions required perfect insight into the state of an entire discipline or posed a fundamental unsolved problem.[93] Thus, the invitation was typically not addressed to young aspirants of science but to the leading savants —Euler, the Bernoullis, Lagrange, d'Alembert, Legendre—who willingly accepted it and the result of their efforts advanced the sciences.

In some cases, there is evidence that a topic of a contest may have been set with someone in mind. In the case of the Institut

[93] Grey, J., *A History of Prizes in Mathematics*. In the Millennium Prize Problems, J. Carlson, A. Jaffe, and A. Wiles, Editors. Providence, RI: American Mathematical Society and Clay Mathematics Institute (2006), p. 6.



17

Experiments with Vibrating Plates

Il nous reste à faire connaître les résultats de l'expérience à l'égard de l'influence qu'a sur les sons l'inégale répartition de l'épaisseur entre les différents points de la lame vibrante. Sophie Germain, 1823-25.

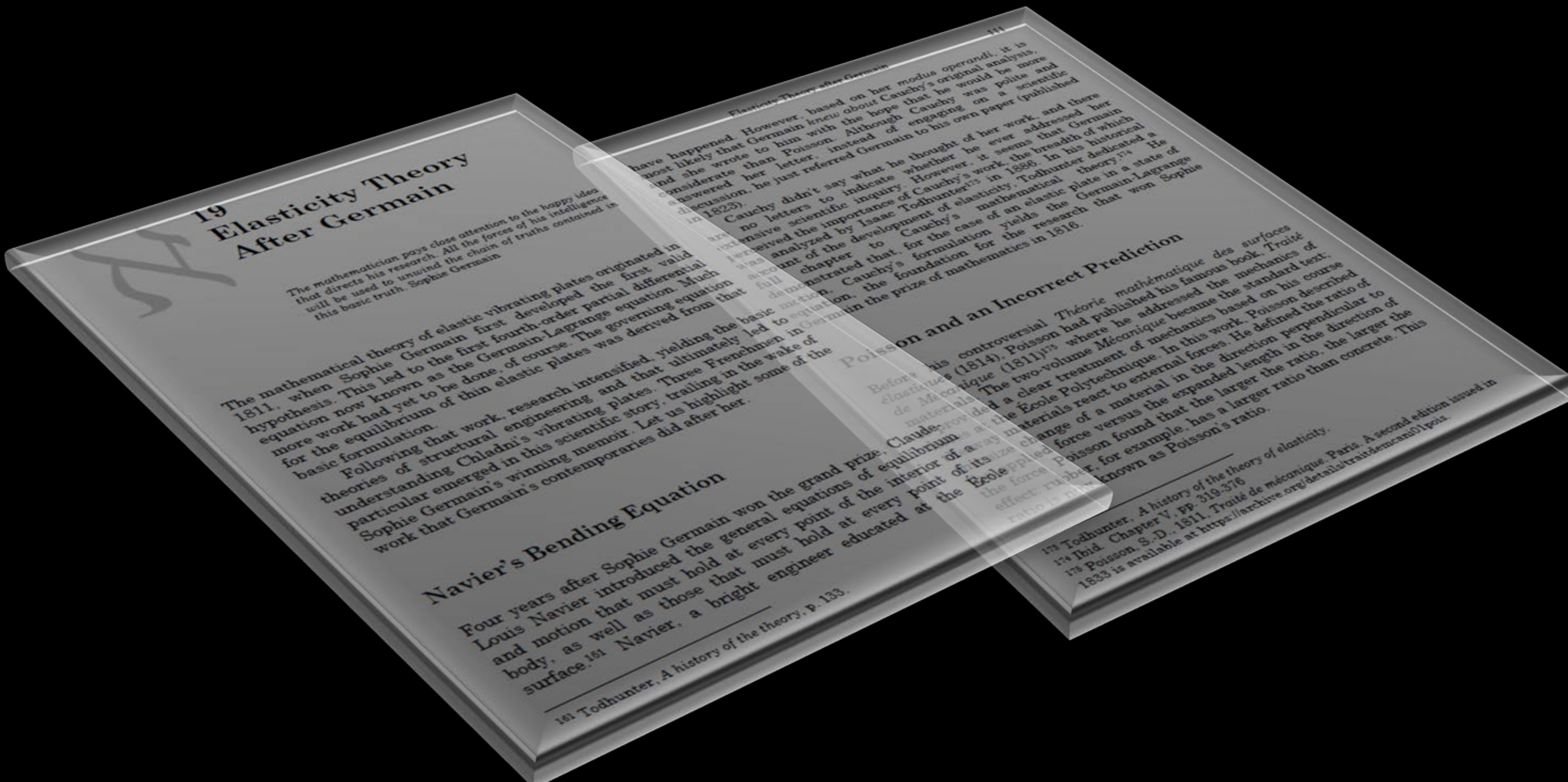
When Sophie Germain attempted to develop a theory for vibrating plates of variable thickness, she was aware of the complexity and importance of the problem. She had to build especial plates to carry out her own experiments. Her memoir of 1825 begins with a review of the relevant literature, citing papers by Euler, Bernoulli, Lagrange, Chladni, Poisson, Navier, Savart, and Italian physicist Giordano Riccati. It is evident that Germain remained abreast of scientific developments in her area of research; she read the papers presented at the Paris Academy of Sciences, especially those by Poisson and Navier, and she provided her own commentaries about their results.[146]

Did Germain realize that she needed the governing fourth-order partial differential equation with variable coefficients in w in order to describe the bending of thin plates with variable thickness? As we know, a closed form solution of such an equation is possible only in very special cases. Today's engineers analyze plates of variable thickness using approximate and numerical methods such as the variational approach (the Ritz method), finite element methods, and the small parameter method.

In the course of her research, Sophie conducted experiments to understand the nature of the vibration and elasticity of her plates, trying to reconcile her hypothesis to the sand patterns in

[146] Germain, S., *Mémoire sur l'emploi de l'épaisseur dans la théorie des surfaces élastiques*, Journal de mathématiques pures et appliquées 3e série, tome 6, 1880, p. S11-64.

Who else contributed to develop the theory of elasticity and vibrations?



19
Elasticity Theory After Germain

The mathematician pays close attention to the happy idea that directs his research. All the forces of his intelligence will be used to unwind the chain of truths contained in this basic truth. Sophie Germain

The mathematical theory of elastic vibrating plates originated in 1811, when Sophie Germain first developed the first valid hypothesis. This led to the first fourth-order partial differential equation now known as the Germain-Lagrange equation. Much more work had yet to be done, of course. The governing equation for the equilibrium of thin elastic plates was derived from that basic formulation.

Following that work, research intensified, yielding the basic theories of structural engineering and that ultimately led to understanding Chladni's vibrating plates. Three Frenchmen in particular emerged in this scientific story, trailing in the wake of Sophie Germain's winning memoir. Let us highlight some of the work that Germain's contemporaries did after her.

Navier's Bending Equation

Four years after Sophie Germain won the grand prize, Claude-Louis Navier introduced the general equations of equilibrium and motion that must hold at every point of the interior of a body, as well as those that must hold at every point of its surface.[161] Navier, a bright engineer educated at the École

[161] Todhunter, *A history of the theory*, p.133.

have happened. However, based on her *modus operandi*, it is most likely that Germain *knew about* Cauchy's original analysis, and she wrote to him with the hope that he would be more considerate than Poisson. Although Cauchy was polite and answered her letter, instead of engaging on a scientific discussion, he just referred Germain to his own paper (published in 1823).

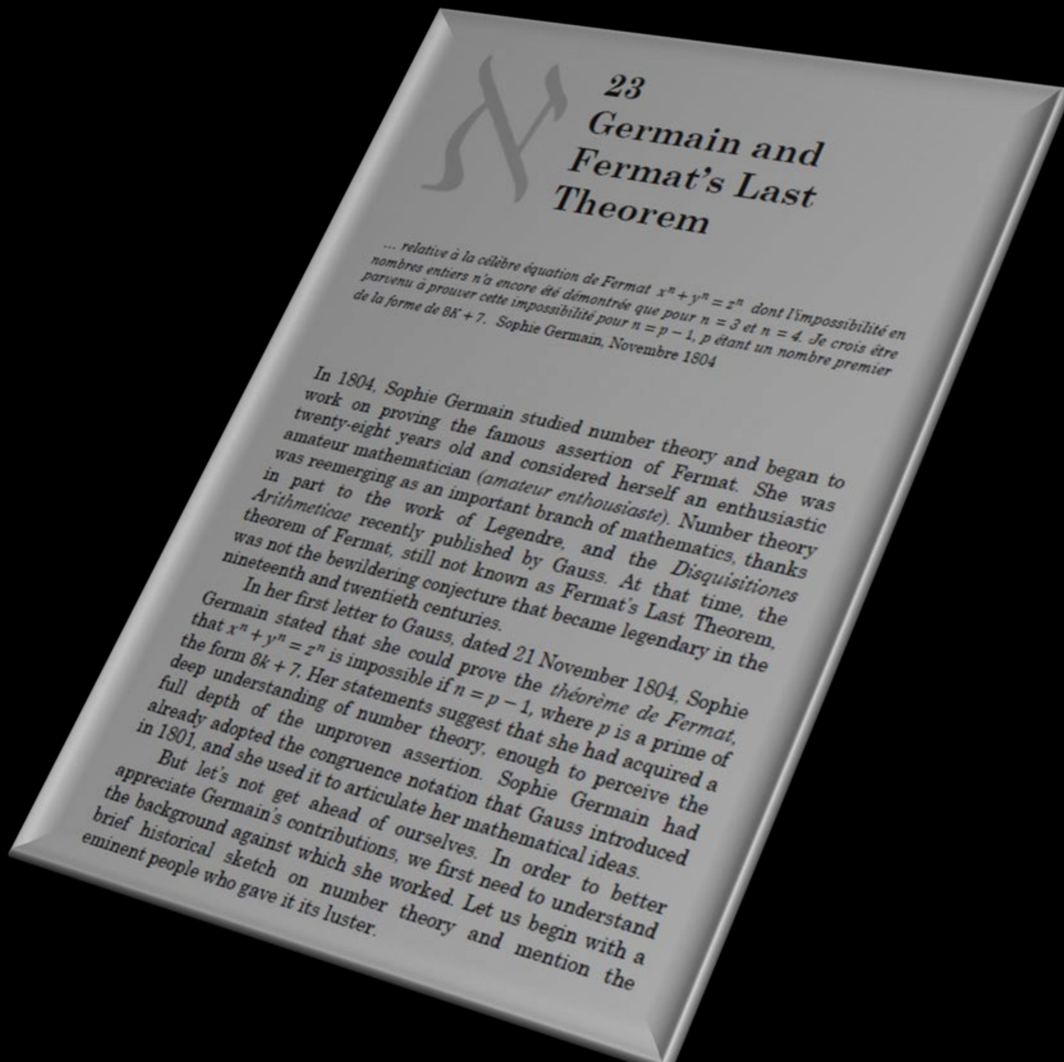
Cauchy didn't say what he thought of her work, and there are no letters to indicate whether he ever addressed her extensive scientific inquiry. However, it seems that Germain perceived the importance of Cauchy's work, the breadth of which was analyzed by Isaac Todhunter[173] in 1886. In his historical account of the development of elasticity, Todhunter dedicated a full chapter to Cauchy's mathematical theory.[174] He demonstrated that, for the case of an elastic plate in a state of motion, Cauchy's formulation yields the Germain-Lagrange equation, the foundation for the research that won Sophie Germain the prize of mathematics in 1816.

Poisson and an Incorrect Prediction

Before this controversial *Théorie mathématique des surfaces élastiques* (1814), Poisson had published his famous book, *Traité de Mécanique* (1811),[175] where he addressed the mechanics of materials. The two-volume *Mécanique* became the standard text; provided a clear treatment of mechanics based on his course at the École Polytechnique. In this work, Poisson described how materials react to external forces. He defined the ratio of change of a material in the direction perpendicular to the force versus the expanded length in the direction of the applied force. Poisson found that the larger the ratio, the larger the effect: rubber, for example, has a larger ratio than concrete. This ratio is now known as Poisson's ratio.

[173] Todhunter, *A history of the theory of elasticity*.
[174] Ibid. Chapter V, pp. 319-376
[175] Poisson, S.D., 1811. *Traité de mécanique*. Paris. A second edition issued in 1833 is available at https://archive.org/details/traitdemecani01pois

What type of mathematical analysis did **Sophie Germain** carry out to **develop her plan to prove Fermat's Last Theorem** ? Who knew about it? How did her theorem became known publicly?



23

Germain and Fermat's Last Theorem

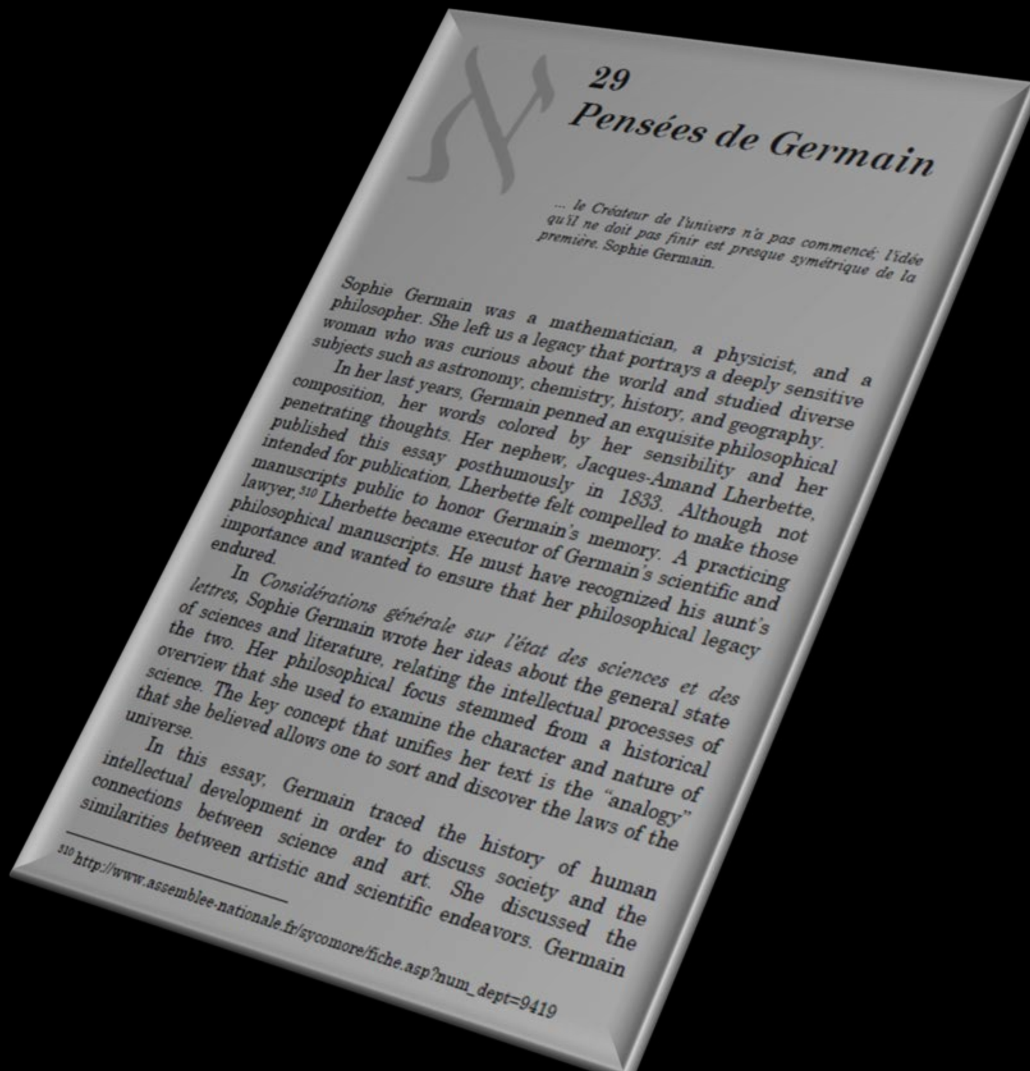
... relative à la célèbre équation de Fermat $x^n + y^n = z^n$ dont l'impossibilité en nombres entiers n'a encore été démontrée que pour $n = 3$ et $n = 4$. Je crois être parvenu à prouver cette impossibilité pour $n = p - 1$, p étant un nombre premier de la forme de $8K + 7$. Sophie Germain, Novembre 1804

In 1804, Sophie Germain studied number theory and began to work on proving the famous assertion of Fermat. She was twenty-eight years old and considered herself an enthusiastic amateur mathematician (*amateur enthousiaste*). Number theory was reemerging as an important branch of mathematics, thanks in part to the work of Legendre, and the *Disquisitiones Arithmeticae* recently published by Gauss. At that time, the theorem of Fermat, still not known as Fermat's Last Theorem, was not the bewildering conjecture that became legendary in the nineteenth and twentieth centuries.

In her first letter to Gauss, dated 21 November 1804, Sophie Germain stated that she could prove the *théorème de Fermat*, that $x^n + y^n = z^n$ is impossible if $n = p - 1$, where p is a prime of the form $8k + 7$. Her statements suggest that she had acquired a deep understanding of number theory, enough to perceive the full depth of the unproven assertion. Sophie Germain had already adopted the congruence notation that Gauss introduced in 1801, and she used it to articulate her mathematical ideas.

But let's not get ahead of ourselves. In order to better appreciate Germain's contributions, we first need to understand the background against which she worked. Let us begin with a brief historical sketch on number theory and mention the eminent people who gave it its luster.

Who was Sophie Germain? What did she think about the pursuit of science and mathematics?



29

Pensées de Germain

... le Créateur de l'univers n'a pas commencé; l'idée qu'il ne doit pas finir est presque symétrique de la première. Sophie Germain.

Sophie Germain was a mathematician, a physicist, and a philosopher. She left us a legacy that portrays a deeply sensitive woman who was curious about the world and studied diverse subjects such as astronomy, chemistry, history, and geography.

In her last years, Germain penned an exquisite philosophical composition, her words colored by her sensibility and her penetrating thoughts. Her nephew, Jacques-Amand Lherbette, published this essay posthumously in 1833. Although not intended for publication, Lherbette felt compelled to make those manuscripts public to honor Germain's memory. A practicing lawyer,[310] Lherbette became executor of Germain's scientific and philosophical manuscripts. He must have recognized his aunt's importance and wanted to ensure that her philosophical legacy endured.

In *Considérations générale sur l'état des sciences et des lettres*, Sophie Germain wrote her ideas about the general state of sciences and literature, relating the intellectual processes of the two. Her philosophical focus stemmed from a historical overview that she used to examine the character and nature of science. The key concept that unifies her text is the "analogy" that she believed allows one to sort and discover the laws of the universe.

In this essay, Germain traced the history of human intellectual development in order to discuss society and the connections between science and art. She discussed the similarities between artistic and scientific endeavors. Germain

[310] http://www.assemblee-nationale.fr/sycomore/fiche.asp?num_dept=9419

Sophie Germain's Contribution

$$x^n + y^n = z^n$$

Sophie Germain was the first and only woman to advance the proof of Fermat's Last Theorem.

Chapter 23 portrays her obsession to find a proof, her theorem, and her relationship with Gauss and Legendre.

Sophie Germain Primes

Given p prime, the number is Sophie Germain prime if $2p + 1$ is also prime.

Let us verify:
$2 \rightarrow 2 \cdot 2 + 1 = 5$ (prime) $\rightarrow 2$ is Germain prime
$3 \rightarrow 2 \cdot 3 + 1 = 7$ (prime) $\rightarrow 3$ is Germain prime
$5 \rightarrow 2 \cdot 5 + 1 = 11$ (prime) $\rightarrow 5$ is Germain prime
$7 \rightarrow 2 \cdot 7 + 1 = 15$ (not prime) $\rightarrow 7$ is *not* Germain prime

While there are 169 prime numbers in the interval [1, 1000], only 37 of those are Sophie Germain primes.

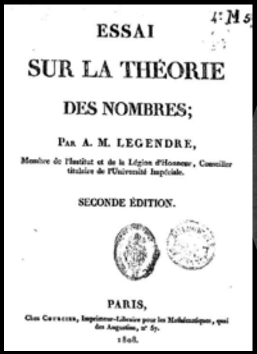
2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191, 233, 239, 251, 281, 293, 359, 419, 431, 443, 491, 509, 593, 641, 653, 659, 683, 719, 743, 761, 809, 911, 953, 1013, 1019, 1031, 1049, 1103, 1223, 1229, 1289, 1409, 1439, 1451, 1481, 1499, 1511, 1559, 1583, 1601, 1733, 1811, 1889, 1901, 1931, 1973, 2003, 2039, 2063, …

How many Sophie Germain are there?

One would conjecture that there exist infinitely many primes p such that $2p + 1$ is also a prime. However, just as Goldbach Conjecture, it has not been proved. To date, the largest Sophie Germain prime is which has 200,701 digits; it was discovered in 2012.

FRIENDS, RIVALS, and MENTORS

Gauss



ESSAI

SUR LA THÉORIE

DES NOMBRES;

PAR A. M. LEGENDRE,

Membre de l'Institut et de la Légion d'Honneur, Conseiller titulaire de l'Université Impériale.

SECONDE ÉDITION.

PARIS,

1808.

Legendre

Lagrange

Fourier

Poisson

Navier

Cauchy

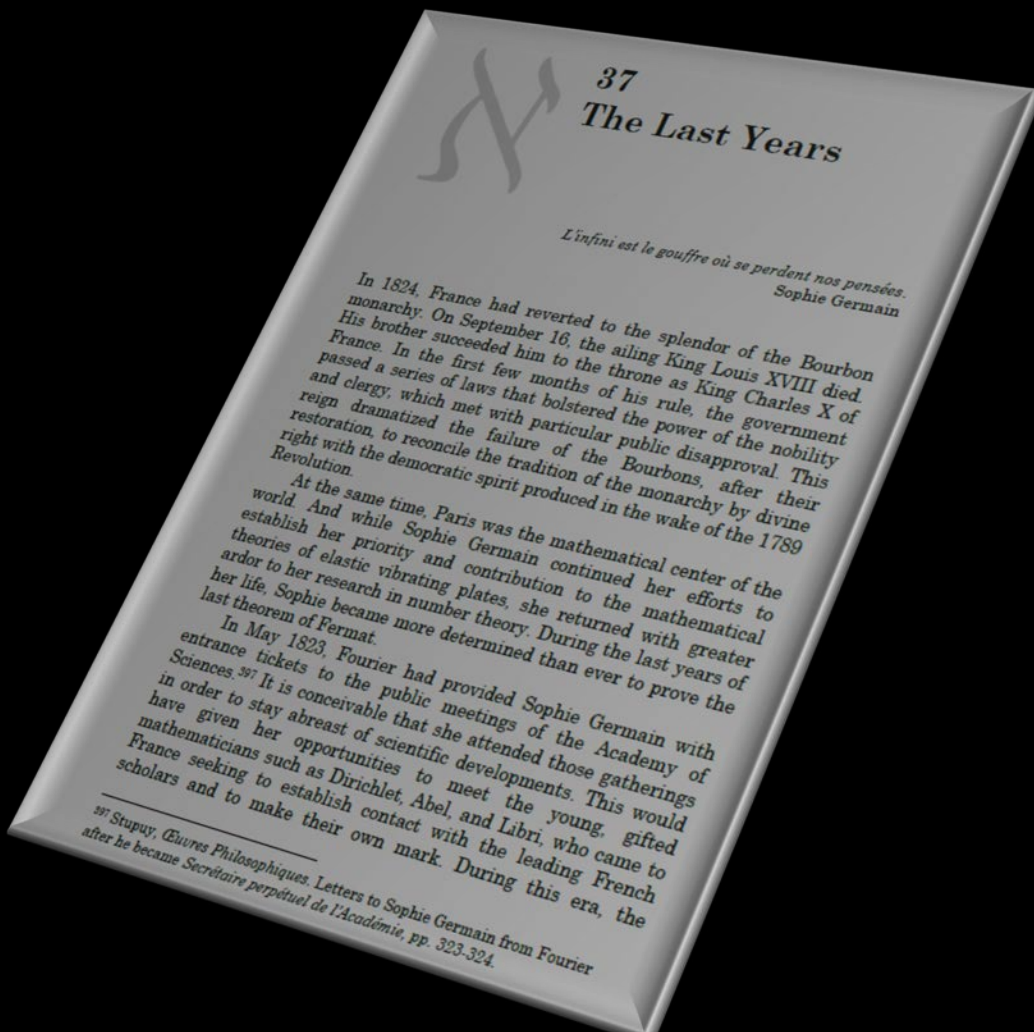
Libri

Sophie Germain Worked, Socialized, and Fought with the best Mathematicians and Scientists of Her Time

Who where her true friends?

Chapter 31 reveals who taught and mentored Sophie Germain, and who snubbed or admired her intellect

How did Sophie Germain spend her last years? Who did she befriend? What events shaped her intellectual world?



37

The Last Years

L'infini est le gouffre où se perdent nos pensées.
Sophie Germain

In 1824, France had reverted to the splendor of the Bourbon monarchy. On September 16, the ailing King Louis XVIII died. His brother succeeded him to the throne as King Charles X of France. In the first few months of his rule, the government passed a series of laws that bolstered the power of the nobility and clergy, which met with particular public disapproval. This reign dramatized the failure of the Bourbons, after their restoration, to reconcile the tradition of the monarchy by divine right with the democratic spirit produced in the wake of the 1789 Revolution.

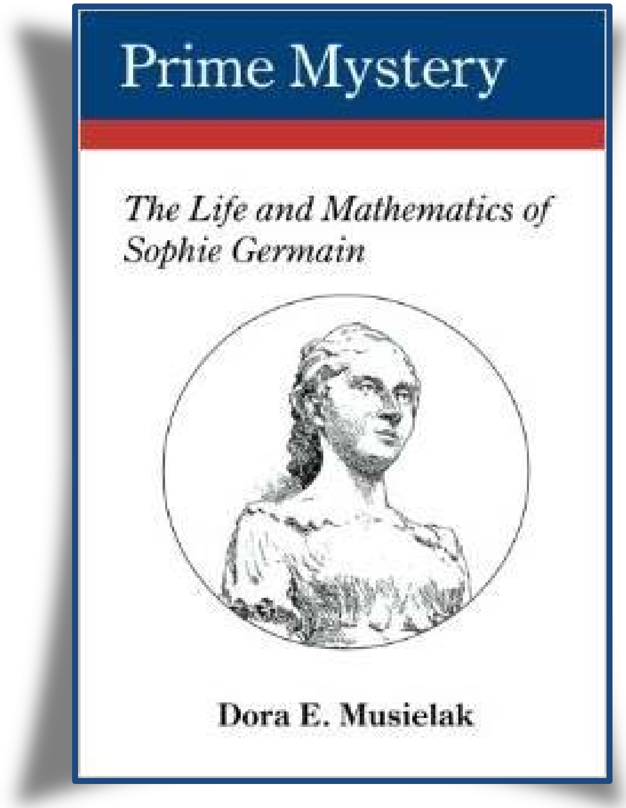
At the same time, Paris was the mathematical center of the world. And while Sophie Germain continued her efforts to establish her priority and contribution to the mathematical theories of elastic vibrating plates, she returned with greater ardor to her research in number theory. During the last years of her life, Sophie became more determined than ever to prove the last theorem of Fermat.

In May 1823, Fourier had provided Sophie Germain with entrance tickets to the public meetings of the Academy of Sciences.[397] It is conceivable that she attended those gatherings in order to stay abreast of scientific developments. This would have given her opportunities to meet the young, gifted mathematicians such as Dirichlet, Abel, and Libri, who came to France seeking to establish contact with the leading French scholars and to make their own mark. During this era, the

[397] Stupuy, *Œuvres Philosophiques*, Letters to Sophie Germain from Fourier after he became *Secrétaire perpétuel de l'Académie*, pp. 323-324.

Prime Mystery: The Life and Mathematics of Sophie Germain paints a rich portrait of the brilliant and complex woman, including the mathematics she developed, her associations with Gauss, Legendre, and other leading researchers, and the tumultuous times in which she lived.

In *Prime Mystery*, author Dora Musielak has done the impossible —she has chronicled Sophie Germain's brilliance through her life and work in mathematics, in a way that is simultaneously informative, comprehensive, and accurate.



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Dora Musielak writes articles on the history of mathematics. She teaches applied mathematics to graduate students of physics and engineering. Musielak is member of the Mathematical Association of America (MAA).

Dora Musielak
Author of ***Sophie's Diary***

Prime Mystery

The Life and Mathematics of Sophie Germain

by **Dora Musielak**
Author of *Sophie's Diary*

In celebration of **Sophie Germain Day**