3315

OPERATIONS RESEARCH I

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STANDARD FORM FOR A LPP

$$\underset{(x_1, \ldots, x_n)}{\text{Maximize}}\ z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le , = , \ge b_1$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le , = , \ge b_m$$

$$x_1, \ldots, x_n \ge 0$$

DIET EXAMPLE

Consider an impecunious student who decides to eat meals consisting of only milk, tuna fish, bread, and spinach. Formulate a linear program to minimize cost while meeting the required daily allowance (RDA) of various nutrients.

DATA

	Gallons of milk x_1	Pounds of Tuna x_2	Loaves of Bread x_3	Pounds of Spinach x_4	RDA
Vit. A	6400	237	0	34000	5000 IU
Vit. C	40	0	0	71	75 mg
Vit. D	540	0	0	0	400 IU
Iron	28	7	13	8	12 mg
Cost	\$3.00	\$2.70	\$1.80	\$2.16	

IU = International Unit; mg= milligram

Minimize $z = 3.00x_1 + 2.70x_2 + 1.80x_3 + 2.16x_4$

s.t.

$6400x_1 + 237x_2 + 0x_3 + 34000x_4$	≥ 5000	vit. A
$40x_1 + 0x_2 + 0x_3 + 71x_4$	≥ 75	vit. C
$540x_1 + 0x_2 + 0x_3 + 0x_4$	≥ 400	vit. D
$28x_1 + 7x_2 + 13x_3 + 8x_4$	≥ 12	iron
x_1, x_2, x_3, x_4	≥ 0	

EXAMPLE

Greenthumb.com, a fertilizer company, wants to make two types of fertilizers: high-nitrogen and all-purpose fertilizers. There are two types of components needed to make these fertilizers. Component 1 consists of 60% nitrogen and 10% phosphorous and costs 20 cents per pound. Component 2 consists of 10% nitrogen and 40% phosphorous and costs 30 cents per pound. The company wants to produce 5000 25-pound bags of high-nitrogen fertilizer and 7000 25-pound bags of all-purpose fertilizer.

Let x_1 denote the amount of component 1 required to make high-nitrogen fertilizer and x_2 denote the amount of component 2 required to make high-nitrogen fertilizer. Similarly let y_1 and y_2 denote the amounts of component 1 and component 2, respectively, required to make all-purpose fertilizer. High-nitrogen fertilizer must contain 40 – 50 % nitrogen by weight and all-purpose fertilizer must contain at most 20 % phosphorous by weight. Greenthumb.com wants to minimize the total cost of producing these fertilizers.

Minimize $z = 0.2(x_1 + y_1) + 0.3(x_2 + y_2)$

s.t.

$$x_1 + x_2 \geq 125{,}000$$

$$y_1 + y_2 \geq 175{,}000$$

$$-0.20x_1 + 0.30x_2 \leq 0$$

$$0.10x_1 - 0.40x_2 \leq 0$$

$$-0.10y_1 + 0.20y_2 \leq 0$$

$$x_1, x_2, y_1, y_2 \geq 0.$$

EXAMPLE

ChemLabs uses raw materials I and II to produce two domestic cleaning solutions A and B. The daily availabilities of raw materials I and II are 160 and 145 units, respectively. One unit of solution A consumes 0.5 unit of raw material I and 0.6 unit of raw material II, and one unit of solution B uses 0.4 unit of raw material I and 0.3 unit of raw material II. The profits per unit of solutions A and B are \$8 and \$10, respectively. The daily demands for solutions A and B are exactly 175 and 200 units, respectively, all of which may not be able to be fulfilled. Find the optimal daily production amounts of A and B.

SOLUTION

Let x be the units of solution A to be made, and let y be the units of solution B to be made.

$$\text{Maximize } z = 8x + 10y$$

s.t.

$$0.5x + 0.4y \le 160 \quad \text{I}$$

$$0.6x + 0.3y \le 145 \quad \text{II}$$

$$x \le 175 \quad \text{A}$$

$$y \le 200 \quad \text{B}$$

$$x, y \ge 0$$

HOMEWORK 1

An industrial recycling center uses two scrap aluminum metals, A and B, to produce a special alloy. Scrap A contains 6% aluminum, 3% silicon, and 4% carbon. Scrap B has 3% aluminum, 6% silicon and 3% carbon. The costs per ton for scraps A and B are \$100 and \$80, respectively. The specifications of the special alloy are as follows:

1. The aluminum content must be at least 3% and at most 6%
2. The silicon content must lie between 3% and 5%
3. The carbon content must be between 3% and 7%

Formulate a linear program that can be used to determine the amounts of scrap A and B that should be used to minimize the cost of creating 1000 tons of the special alloy.

GRAPHICAL SOLUTION OF LINEAR PROGRAMMING

Maximize $z = 3x_1+5x_2$

subject to

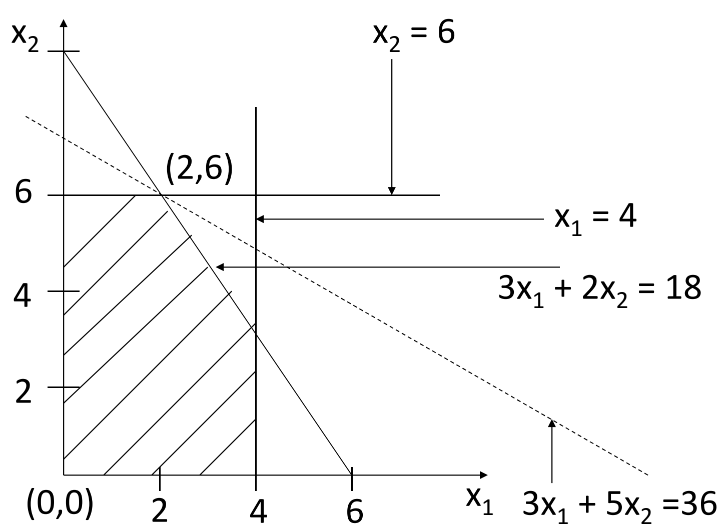
$3x_1 + 2x_2 \leq 18$

$x_1 \leq 4$

$x_2 \leq 6$

$x_1, x_2 \geq 0$

GRAPHICAL SOLUTION



DEFINITIONS

FEASIBLE SOLUTION: an $(x_1,\ldots x_n)$ satisfying all the constraints

OPTIMAL SOLUTION: a best feasible solution

EXTREME POINT: a corner of the convex set of feasible solutions

PROPERTIES

PROPERTY 1:
The set of feasible solutions to a linear programming problem is a convex set.

PROPERTY 2:
If there exists an optimal solution, then at least one extreme point is optimal.

PROPERTY 3:
There are only a finite number of extreme points.

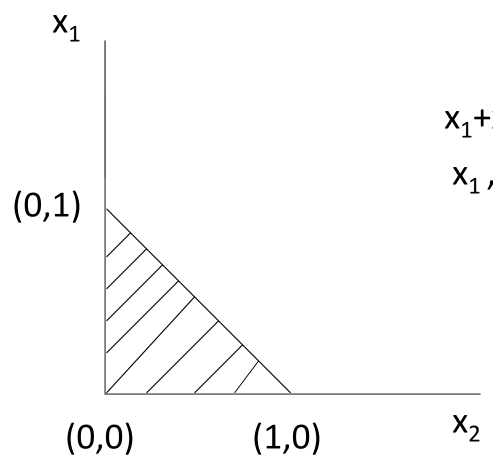
Before giving properties 4 and 5, rewrite the problem by adding slack variables. A slack variable is added to make an inequality into an equation. By adding slack variables, the problem becomes

Maximize $z = 3x_1+5x_2$

subject to

$$
\begin{aligned}
3x_1 + 2x_2 + x_3 \qquad &= 18 \\
x_1 \qquad + x_4 \qquad &= 4 \\
x_2 \qquad + x_5 &= 6 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0.
\end{aligned}
$$

GRAPHICAL INTERPRETATION

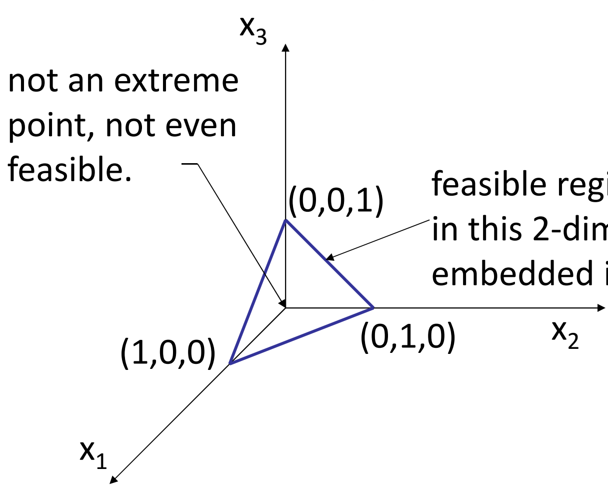


$x_1+x_2 \leq 1$

$x_1, x_2 \geq 0$

$x_1+x_2+x_3 = 1$

$x_1, x_2, x_3 \geq 0$



BASIC SOLUTIONS

For m equations and n unknown variables (m<n), solve for m variables in terms of (n-m) set to 0. Consider

$$x_1+x_2+x_3=2$$
$$2x_1-x_2+x_3=1.$$

Pick x_3, set it to zero, and solve for the others.

$$x_1+x_2=2$$
$$2x_1-x_2=1$$

$x_1=1$, $x_2=1$ are basic variables.
$x_3=0$ is a non-basic variable.

EXAMPLE

$$x_1+x_2+x_3=2$$
$$2\,x_1-x_2+x_3=1$$

Pick x_2, set it to zero, and solve for the others.

$$x_1+x_3=2$$
$$2x_1+x_3=1$$

$x_1=-1$, $x_3=3$ are basic variables.

$x_2=0$ is a non-basic variable.

BASIC FEASIBLE SOLUTION

A basic solution to the constraints (omitting non-negativity) of LPP with slacks and surplus variables with all basic variables ≥ 0.

In the previous examples, Example 1 yielded a BFS while Example 2 did not.

DEGENERATE BFS

A basic feasible solution where at least one basic variable is 0.

Example 1:

$$x_1 + x_2 + x_3 = 1$$

$$2x_1 + x_2 + x_3 = 2$$

Pick $x_3 = 0$ as nonbasic and x_1, x_2 as basic. Solve

$$x_1 + x_2 = 1$$

$$2x_1 + x_2 = 2$$

to obtain $x_1 = 1$, $x_2 = 0$ for the basic variables.

Example 2:

$$x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 + 3x_3 = 2$$

Arbitrarily chose x_3 as the nonbasic variable to give

$$x_1 + x_2 = 1$$

$$x_1 + x_2 = 2.$$

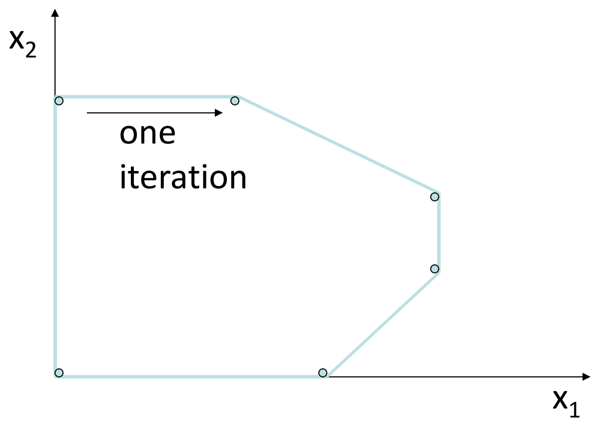
Note the contradiction. Thus there is no basic solution with $x_3 = 0$.

Property 4:

With slacks as needed to make all constraints equality, there is a one-to-one correspondence between the BFS of these equality constraints and the extreme points of the original constraints.

Property 5:

Changing one BV in a BFS moves to an adjacent extreme point.



EXAMPLE

Maximize $z = 3x_1 + 5x_2$

s.t.

$3x_1 + 2x_2 \leq 18$

$x_1 \leq 4$

$x_2 \leq 6$

$x_1, x_2 \geq 0$

Rewrite to get

Maximize $z = 3x_1+5x_2$

subject to

$$
\begin{aligned}
3x_1 + 2x_2 + x_3 \quad\quad &= 18,\\
x_1 \quad\quad + x_4 \quad &= 4,\\
x_2 \quad\quad + x_5 &= 6,\\
x_1,x_2,x_3,x_4,x_5 &\geq 0
\end{aligned}
$$

Rewrite to give

$$
\begin{aligned}
z - 3x_1 - 5x_2 \quad\quad &= 0\\
3x_1 + 2x_2 + x_3 \quad &= 18\\
x_1 \quad\quad + x_4 \quad &= 4\\
x_2 \quad\quad + x_5 &= 6.
\end{aligned}
$$

Note: m=3 , n=5 => 10 possible BFS's

A BFS is found using the last 3 equations

while keeping track of z using the first equation.

Initial BFS:

$x_3 = 18,\ x_4 = 4,\ x_5 = 6$ (BV)

$x_1 = 0,\ x_2 = 0$ (NBV)

and $z = 0$

Note that making x_2 positive increases z the most per unit.

Hence, make x_2 a BV and keep x_1=0 a NBV.

Next, rewrite the last 3 equations to give

$$x_3 = 18 - 3x_1 - 2x_2$$
$$x_4 = 4 - x_1$$
$$x_5 = 6 - x_2.$$

Now

$x_3 = 18 - 2x_2 \geq 0 \Rightarrow x_2 \leq 9$

$x_4 = 4$

$x_5 = 6 - x_2 \geq 0 \Rightarrow x_2 \leq 6$

So x_2 enters and x_5 leaves.

The new BFS is adjacent to the earlier one.

$$z - 3x_1 - 5x_2 \qquad\qquad = 0$$

$$3x_1 + 2x_2 + x_3 \qquad = 18$$

$$x_1 \qquad + x_4 \qquad = 4$$

$$x_2 \qquad + x_5 = 6$$

Multiply 4th equation by 5 and add to z row and multiply 4th equation by -2 and add to 2nd equation.

$$z - 3x_1 \qquad\qquad + 5x_5 = 30$$

$$3x_1 \qquad + x_3 \quad - 2x_5 = 6$$

$$x_1 \qquad + x_4 \qquad = 4$$

$$x_2 \qquad + x_5 = 6$$

It follows that

$$z \qquad + x_3 \qquad + 3x_5 = 36$$

$$x_1 \quad + 1/3x_3 \qquad -2/3x_5 = 2$$

$$-1/3x_3 \ + x_4 \ + 2/3x_5 = 2$$

$$x_2 \qquad\qquad + x_5 = 6.$$

All of the negative variables have been removed from the objective function meaning that it has been optimized. Thus z = 36, $x_1 = 2$, $x_2 = 6$.

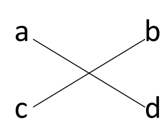
TABLEAU APPROACH

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-3	-5	0	0	0	0
x_3	0	3	2	1	0	0	18
x_4	0	1	0	0	1	0	4
x_5	0	0	1	0	0	1	6

TABLEAU ONE

GAUSS – JORDAN ELIMINATION

The pivot element is the intersection of the leaving and entering variables in the tableau.



Let a = pivot element.
Then the new element d_{new} = (ad – bc) / a .

SIMPLEX ALGORITHM

1. Must have all RHS (not z) $\geq$ 0.
2. Entering variable is most negative element in top row.
3. Leaving variable is found by finding the smallest ratio of the RHS to positive elements of pivot column. (If no positive elements, the problem is unbounded.)
4. Form new tableau using Gauss-Jordan elimination.
5. If elements in top row $\geq$ 0, stop. Otherwise, go to 2.

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-3	0	0	0	5	30
x_3	0	3	0	1	0	-2	6
x_4	0	1	0	0	1	0	4
x_2	0	0	1	0	0	1	6

TABLEAU TWO

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	1	0	3	36
x_1	0	1	0	1/3	0	-2/3	2
x_4	0	0	0	-1/3	1	2/3	2
x_2	0	0	1	0	0	1	6

FINAL TABLEAU

SOLUTION

$z = 36$

$x_1 = 2$

$x_2 = 6$

EXAMPLE

Maximize $z = 2x_1+3x_2$

subject to

$$x_1 + x_2 \leq 3$$

$$x_1 - x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Adding slack variables, we get

$$z - 2x_1 - 3x_2 = 0$$

$$x_1 + x_2 + x_3 = 3$$

$$x_1 - x_2 + x_4 = 1$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

TABLEAU

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	-2	-3	0	0	0
x_3	0	1	1	1	0	3
x_4	0	1	-1	0	1	1
z	1	1	0	3	0	9
x_2	0	1	1	1	0	3
x_4	0	2	0	1	1	4

SOLUTION

$z = 9$

$x_1 = 0$

$x_2 = 3$

HOMEWORK 2

Maximize $z = 2x_1+5x_2$

subject to

$x_1 + x_2 \leq 12$

$3x_1+ x_2 \leq 18$

$x_1,x_2 \geq 0$

Solve using the simplex method by the tableau approach.

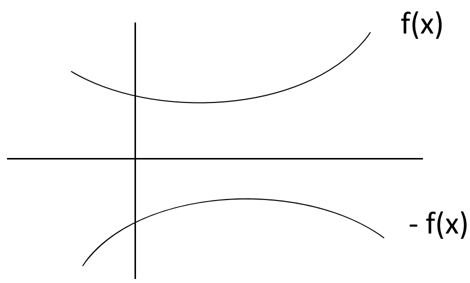
HOMEWORK 3-8

Six linear programming problems solved by Lingo are due February 20 at the beginning of class. Some form of computer printout of the solution is required. In all cases, simply apply Tora to the mathematical problem and ignore the directions. In the tenth edition of Taha, these problems are:

page 122 - #3-32
page 126 - # 3-50
page 126 - # 3-52
page 129 - #3-59
page 129 - #3-62
page 363 - #9-60.

COMPLICATIONS

1. Minimization



Thus Min f(x) = Max -f(x), so just maximize -z

Alternately, note that in maximization problem the most negative value is chosen from the top row of the tableau. For a minimization problem the most positive value is selected from the top row of the table. All other steps are the same for both maximization and minimization problems.

Minimize $z = -2x_1 + 5x_2$

subject to

$x_1 + x_2 \leq 12$

$3x_1 + x_2 \leq 18$

$x_1, x_2 \geq 0.$

Add slack variables to get

Minimize $z = -2x_1 + 5x_2$

subject to

$x_1 + x_2 + x_3 \quad = 12$

$3x_1 + x_2 \quad + x_4 = 18$

$x_1, x_2, x_3, x_4 \geq 0.$

BV	z	x_1 ↓	x_2	x_3	x_4	RHS	
z	1	2	-5	0	0	0	
x_3	0	1	1	1	0	12	
x_4	0	(3)	1	0	1	18	←
z	1	0	-17/3	0	-2/3	-12	
x_3	0	0	2/3	1	-1/3	6	
x_1	0	1	1/3	0	1/3	6	

Since there is no other positive value in the top row of the tableau, the objective function has been minimized with

$$z = -12$$
$$x_1 = 6$$
$$x_2 = 0 .$$

HOMEWORK 9

Minimize $z = 2x_1 - 4x_2$

s.t.

$$x_1 + x_2 \leq 10$$
$$2x_1 + x_2 \leq 16$$
$$x_1, x_2 \geq 0$$

2. Unrestricted variables

$$\text{Maximize } z = -2x_1 + x_2$$
subject to
$$x_1 + x_2 \leq 10$$
$$2x_1 + x_2 \leq 16$$
$$x_1 \geq 0$$
$$x_2 \text{ UR}$$

Let $x_2 = x_2^+ - x_2^-$, where $x_2^+, x_2^- \geq 0$. These new variables cannot be basic variables at the same time. At least one of them will be zero, or both of them will be zero.

Rewrite the problem as
$$\text{Maximize } z = -2x_1 + (x_2^+ - x_2^-)$$
subject to
$$x_1 + (x_2^+ - x_2^-) \leq 10$$
$$2x_1 + (x_2^+ - x_2^-) \leq 16$$
$$x_1, x_2^+, x_2^- \geq 0 .$$

Add slack variables to give
$$\text{Maximize } z = -2x_1 + x_2^+ - x_2^-$$
subject to
$$x_1 + x_2^+ - x_2^- + x_3 = 10$$
$$2x_1 + x_2^+ - x_2^- + x_4 = 16$$
$$x_1, x_2^+, x_2^-, x_3, x_4 \geq 0 .$$

Solve as usual and get $x_2 = (x_2^+ - x_2^-)$ afterwards.

HOMEWORK 10

Maximize $z = -2x_1 - 4x_2$

s.t.

$x_1 + 3x_2 \leq 10$

$2x_1 + x_2 \leq 16$

$x_1 \geq 0$, x_2 UR

3. Tie for Entering Variable

The top row of the maximization problem is shown below. It is seen that coefficients of x_1 and x_2 are equal. Break the tie arbitrarily. It doesn't make difference which variable x_1 or x_2 is chosen to enter.

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	-1	-1	0	0	0

4. Tie for leaving variable

In the maximization tableau below the values are same for both x_3 and x_4. Break the tie arbitrarily.

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	-2	0	0	12
x_4	0	1	1	0	1	12
x_3	0	0	1	1	0	12

When there is a tie for leaving variable, the next tableau has a BFS with a 0. Usually this has no effect. Cycling is possible, but there are ways to overcome it. We will discuss this later.

5. Alternate optimal solutions

The final iteration of the maximization problem is shown below in the tableau. If a non-basic variable has zero coefficient in the top row, it means there are alternate optimal solutions. Here x_4 could be entered without changing z.

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	4	0	36
x_1	0	1	0	3	2	12
x_2	0	0	1	2	3	12

6. **No feasible Solution - related to 7 below.**

7. **Initial basic feasible solution**

   To begin a problem, make every constraint into an equality (hopefully with a nonnegative rhs) by adding slack or subtracting surplus variables. If it has $\leq$ sign, add a slack variable. If it has a $\geq$ sign, subtract a surplus variable. For every such constraint without an obvious basic variable, add an artificial variable.

Maximize $z = x_1 + 2x_2 + 3x_3$

subject to,

$$x_1 + x_2 + x_3 = 12$$
$$2x_1 + x_2 - x_3 \geq 10$$
$$3x_1 - x_2 + 2x_3 \leq 18$$
$$x_1, x_2, x_3 \geq 0 .$$

Adding the slack variables to the constraints

$$x_1 + x_2 + x_3 = 12$$
$$2x_1 + x_2 - x_3 - x_4 = 10$$
$$3x_1 - x_2 + 2x_3 + x_5 = 18$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0 .$$

Constraint 1 doesn't have an obvious variable that can be selected as basic variable. The basic variable in constraint 2 is negative. Therefore for these 2 constraints we add artificial variables.

$$x_1 + x_2 + x_3 + x_6 = 12$$

$$2x_1 + x_2 - x_3 - x_4 + x_7 = 10$$

$$3x_1 - x_2 + 2x_3 + x_5 = 18$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0,$$

where x_6, x_7 are artificial variables.

BIG-M METHOD

In the Big M method modify the objective function penalizing artificial variable with M, where positive M is a very big number.

Maximize $z = x_1 + 2x_2 + 3x_3 - Mx_6 - Mx_7$

subject to

$$x_1 + x_2 + x_3 + x_6 = 12$$

$$2x_1 + x_2 - x_3 - x_4 + x_7 = 10$$

$$3x_1 - x_2 + 2x_3 + x_5 = 18$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

BIG-M METHOD

- Solve problem with artificial variables
- Two cases for solution:
  - All artificial variables become non-basic (zero) and z has no "M penality" if and only if the original problem is feasible.
  - At least one artificial variable is basic (positive) and z has an "M penality" in every optimal solution if and only if the original problem is infeasible.

EXAMPLE OF CASE 1

Maximize $z = 2x_1 + x_2$

s.t.

$x_1 + x_2 \leq 10$

$x_1 + 2x_2 \geq 16$

$x_1, x_2 \geq 0.$

Maximize $z = 2x_1 + x_2 - Mx_5$

s.t.

$x_1 + x_2 + x_3 = 10$

$x_1 + 2x_2 - x_4 + x_5 = 16$

$x_1, x_2, x_3, x_4, x_5 \geq 0,$

where x_5 is an artificial variable.

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-2	-1	0	0	M	0
x_3	0	1	1	1	0	0	10
x_5	0	1	2	0	-1	1	16

Making the coefficient of the basic variable 0 in the first row.

BV	z	x_1	x_2 ↓	x_3	x_4	x_5	RHS
z	1	-2-M	-1-2M	0	M	0	-16M
x_3	0	1	1	1	0	0	10
x_5	0	1	(2)	0	-1	1	16 ←

BV	z	x_1 ↓	x_2	X_3	x_4	x_5	RHS
z	1	- 3/2	0	0	-1/2	1/2+ M	8
x_3	0	(1/2)	0	1	1/2	- 1/2	2 ←
x_2	0	1/2	1	0	- 1/2	1/2	8

x_5 left basis so original problem is feasible. Yay!

B.V.	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	3	1	M-1	14
x_1	0	1	0	2	1	-1	4
x_2	0	0	1	-1	-1	1	6

All the coefficients of the x's in the top row are non-negative, so we have the optimal solution z =14, x_1 = 4 and x_2 = 6.

EXAMPLE OF CASE 2

Maximize $z = 2x_1 + x_2$

s.t.

$x_1 + 2x_2 \leq 10$

$x_1 + 2x_2 \geq 16$

$x_1, x_2 \geq 0.$

Maximize $z = 2x_1 + x_2 - Mx_5$

s.t.

$x_1 + 2x_2 + x_3 = 10$

$x_1 + 2x_2 - x_4 + x_5 = 16$

$x_1, x_2, x_3, x_4, x_5 \geq 0,$

where x_5 is an artificial variable.

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-2	-1	0	0	M	0
x_3	0	1	2	1	0	0	10
x_5	0	1	2	0	-1	1	16

Making the coefficient of the basic variable 0 in the first row.

BV	z	x_1	x_2 ↓	x_3	x_4	x_5	RHS
z	1	-2-M	-1-2M	0	M	0	-16M
x_3	0	1	(2)	1	0	0	10 ←
x_5	0	1	2	0	-1	1	16

BV	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	- 3/2	0	1/2+ M	M	0	5-6M
x_2	0	(1/2)	1	1/2	0	0	5
x_5	0	0	0	-1	- 1	1	6

B.V.	z	x_1	X_2	x_3	x_4	x_5	RHS
z	1	0	3	M+2	M	0	20-6M
x_1	0	1	2	1	0	0	10
x_5	0	0	0	-1	-1	1	6

Since all the coefficients of the x's in the top row are non-negative, the solution is z =20-6M, x_1 = 10 and x_5 = 6. But since x_5 is artificial and positive z has a -6M penalty, the original problem is infeasible.

HOMEWORK 11

Maximize $z = x_1 + 3x_2$

s.t.

$x_1 + x_2 \leq 8$

$4x_1 + x_2 \geq 16$

$x_1, x_2 \geq 0.$

8. Unbounded problem

Consider

$$\text{Max } z = x_1$$
$$\text{s.t.}$$
$$x_1 \geq 0.$$

This problem is considered unbounded. Any LP has one of the following:

- No feasible solution (LP is considered *infeasible*)
- An unbounded objective (LP is considered *unbounded*)
- An optimal solution

EXAMPLE

Maximize $z = 2x_1$

s.t.

$-x_1 \leq 10$

$x_1 \geq 0.$

BV	z	x_1	x_2	RHS
z	1	-2	0	0
x_2	0	-1	1	10

Since all the coefficients in the x_1 column are negative, there is no leaving variable. Hence, x_1 and z can be increased indefinitely, and the problem is unbounded.

DUALITY

Primal Problem P:

Maximize $z = \sum_{j=1}^{n} c_j x_j$

s.t.

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$

$x_1, \ldots, x_n \geq 0$

Dual Problem D:

Minimize $w = \sum_{i=1}^{m} b_i y_i$

s.t.

$a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \geq c_1$

$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \geq c_n$

$y_1, \ldots, y_m \geq 0$

DUALITY RELATIONSHIPS

1. The dual of the dual is the primal.

2. For any feasible $(x_1,\ldots,x_n)$ for P and any feasible $(y_1,\ldots,y_m)$ for D ,then $z \leq w$.

3. For any optimal $(x_1^*,\ldots,x_n^*)$ for P and any optimal $(y_1^*,\ldots,y_m^*)$ for D, then $z^* = w^*$. Also, if $z^* = w^*$, then $(x_1^*,\ldots,x_n^*)$ and $(y_1^*,\ldots,y_m^*)$ are optimal.

4. Primal unbounded => dual infeasible
   Primal infeasible => dual unbounded or dual infeasible

   (See table)

Primal Dual States

Dual	Primal		
	Infeasible	Unbounded	Optimal
Infeasible	Rare	X	
Unbounded	X		
Optimal			☺

ANY DUAL

Max	Min
$\leq$ Constraint $\geq$ Constraint $=$ Constraint	≥ 0 Variable ≤ 0 Variable UR Variable
≥ 0 Variable ≤ 0 Variable UR Variable	$\geq$ Constraint $\leq$ Constraint $=$ Constraint

EXAMPLE

P. Minimize $z = 2x_1 + x_2$

s.t.

$$-x_1 + 3x_2 \leq -2$$
$$x_1 - x_2 = 6$$

x_1 UR, $x_2 \leq 0$

D. Maximize $w = -2y_1 + 6y_2$

s.t.

$$-y_1 + y_2 = 2$$
$$3y_1 - y_2 \geq 1$$

$$y_1 \leq 0,\ y_2 \text{ UR}$$

DUAL SIMPLEX

The dual simplex involves solving the dual on the primal tableau with appropriate – signs.

1. Must have optimality criterion (nonnegative coefficients for max problem) satisfied in top row.
2. Leaving variable is most negative RHS excluding the z row.
3. Entering variable is found by finding the smallest ratio of the absolute value of elements in top row (excluding RHS) to negative elements of pivot row. (If no negative elements, problem is infeasible.)
4. Form a new tableau as before.
5. If all RHS (not z) $\geq$ 0, stop. Otherwise, go to 2.

EXAMPLE

Minimize $z = 2x_1 + x_2$

s.t.

$x_1 + x_2 \geq 4$

$5x_1 + 3x_2 \geq 15$

$x_1, x_2 \geq 0.$

Rewriting as equality constraints,

Minimize $z = 2x_1 + x_2$

s.t.

$-x_1 - x_2 + x_3 = -4$

$-5x_1 - 3x_2 + x_4 = -15$

$x_1, x_2, x_3, x_4 \geq 0.$

DUAL SIMPLEX

BV	z	x_1	x_2 ↓	x_3	x_4	RHS
z	1	-2	-1	0	0	0
x_3	0	-1	-1	1	0	-4
x_4	0	-5	(-3)	0	1	-15 ←
z	1	-1/3	0	0	-1/3	5
x_3	0	2/3	0	1	-1/3	1
x_2	0	5/3	1	0	-1/3	5

HOMEWORK 12

Maximize $z = -3x_1 - 2x_2$

s.t

$x_1 + x_2 \geq 4$

$3x_1 + 5x_2 \geq 15$

$x_1, x_2 \geq 0.$

APPLICATION

The dual simplex may be used to solve problems in which one or more constraints are added after an optimal solution is obtained to the original problem.

Maximize $z = 3x_1 + 2x_2$

s.t

$x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0.$

With a slack variable, the problem becomes

$$\text{Maximize } z = 3x_1 + 2x_2$$

s.t

$$x_1 + x_2 + x_3 = 4,$$

$$x_1, x_2, x_3 \geq 0.$$

BV	z	x_1	x_2	x_3		RHS
z	1	-3	-2	0		0
x_3	0	1	1	1		4

BV	z	x_1	x_2	x_3		RHS
z	1	0	1	3		12
x_1	0	1	1	1		4

Now add the constraint

$$x_1 \leq 2$$

to the original problem. With a slack variable $x_4 \geq 0$, this constraint becomes

$$x_1 + x_4 = 2.$$

Add this equation to the tableau and proceed using the dual simplex.

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	1	3	0	12
x_1	0	1	1	1	0	4
x_4	0	1	0	0	1	2

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	1	3	0	12
x_1	0	1	1	1	0	4
x_4	0	0	-1	-1	1	-2

Dual Simplex

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	2	1	10
x_1	0	1	0	0	1	2
x_2	0	0	1	1	-1	2

This tableau yields the new optimal solution.

HOMEWORK 13

Minimize $z = -2x_1 + 5x_2$

s.t

$3x_1 + 4x_2 \leq 12$

$x_1, x_2 \geq 0.$

After solving this problem, add the constraint

$x_1 + 2x_2 \geq 6.$

INTEGER PROGRAMMING

Maximize $z = 3x_1 + x_2$

s.t.

$x_1 + x_2 \leq 4$

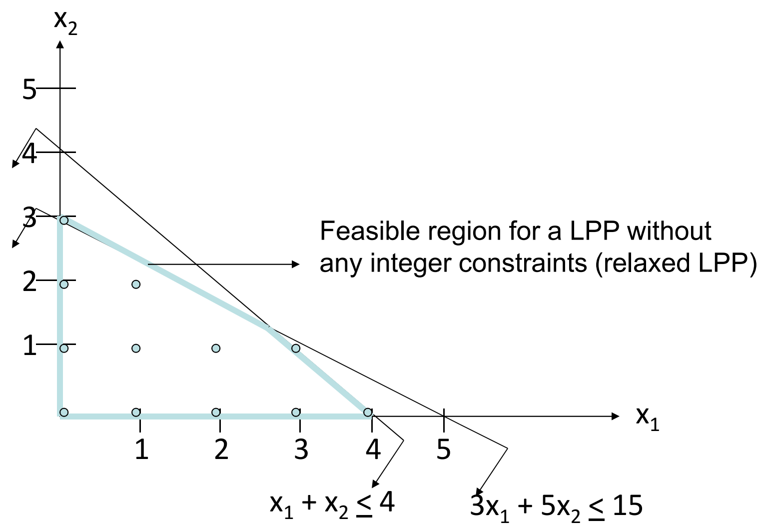
$3x_1 + 5x_2 \leq 15$

$x_1, x_2 \geq 0$

x_1, x_2 integers.

If x_1, x_2 are integers, then the problem is a pure integer programming problem. If only x_1 or x_2 is an integer, then the problem is a mixed integer programming problem.

The feasible region is given by the dots below.



Integer programming problems are usually solved by branch and bound algorithms. There are other methods.

EXAMPLE

$$\text{Max } z = 3x_1 + x_2$$

s.t.

$$4x_1 + 3x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

$$x_1, x_2 \text{ integer.}$$

The constraints become

$$4x_1 + 3x_2 + x_3 = 10$$

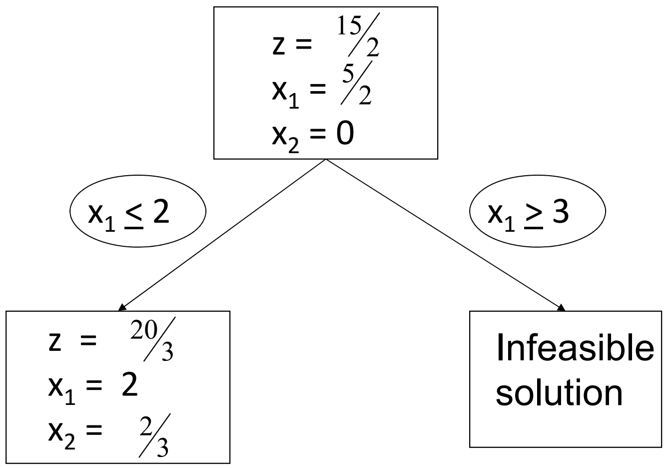
$$x_1, x_2, x_3 \geq 0$$

$$x_1, x_2 \text{ integer.}$$

Step 1: Solve relaxed LPP (without integer restrictions)

BV	z	x_1	x_2	x_3		RHS
z	1	-3	-1	0		0
x_3	0	4	3	1		10
z	1	0	$\frac{5}{4}$	$\frac{3}{4}$		$\frac{15}{2}$
x_1	0	1	$\frac{3}{4}$	$\frac{1}{4}$		$\frac{5}{2}$

Step 2: Use branch and bound



Use relaxed LP after adding x_4 for $x_1 \geq 3$ branch.

$$x_1 - x_4 = 3$$
$$-x_1 + x_4 = -3$$

Add x_4 to the previous table

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	$\frac{5}{4}$	¾	0	$\frac{15}{2}$
x_1	0	1	¾	¼	0	$\frac{5}{2}$
x_4	0	-1	0	0	1	-3

Adding row constraints x_1 and x_4

x_4	0	0	¾	¼	1	-1/2

Dual is unbounded => Primal is infeasible

After adding x_4 for $x_1 \leq 2$, we obtain

$$x_1 + x_4 = 2$$

Add this equality with bv x_4 to the optimal table.

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	$5/4$	$3/4$	0	$15/2$
x_1	0	1	$3/4$	$1/4$	0	$5/2$
x_4	0	1	0	0	1	2

Subtract the x_1 row from the x_4 row to get a new x_4 row.

x_4	0	0	$-3/4$	$-1/4$	1	$-1/2$

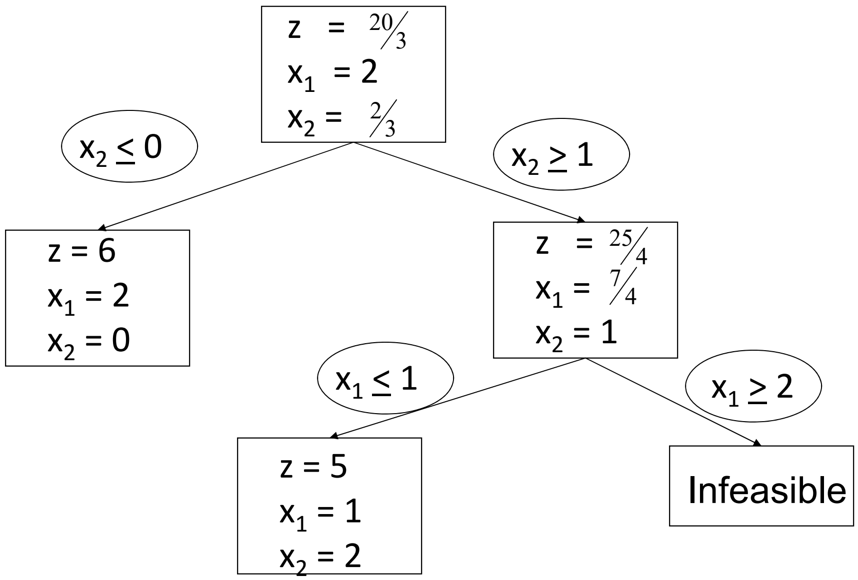
Now x_2 enters and x_4 leaves.

z	1	0	0	$1/3$	$5/3$	$20/3$
x_1	0	1	0	0	1	2
x_2	0	0	1	$1/3$	$-4/3$	$2/3$

<u>Fathoming Rules</u>:

1. Infeasible
2. All x_i integer that supposed to be
3. A value of z no better than some z for feasible x_i to original integer restrictions.

Using branch and bound,



HOMEWORK 14

Max $z = 3x_1 + 2x_2$

s.t.

$6x_1 + 5x_2 \leq 21$

$x_1, x_2 \geq 0$

x_1, x_2 integer

MIXED IPP EXAMPLE

$$\text{Max } z = 4x_1 - 2x_2 + 7x_3 - x_4$$

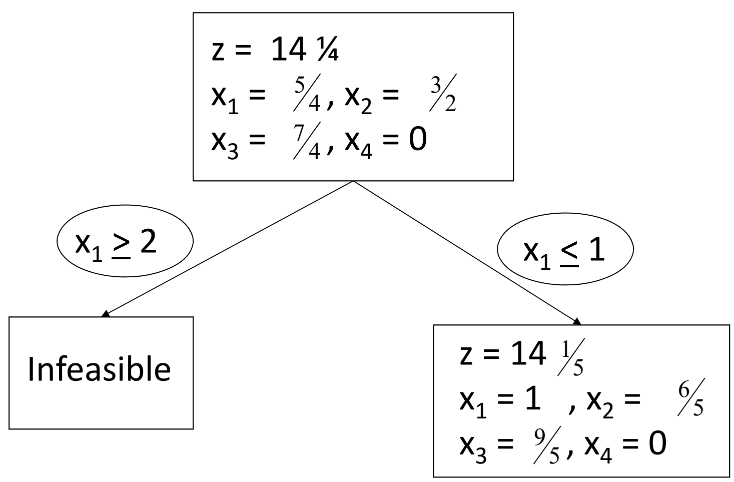
s.t.

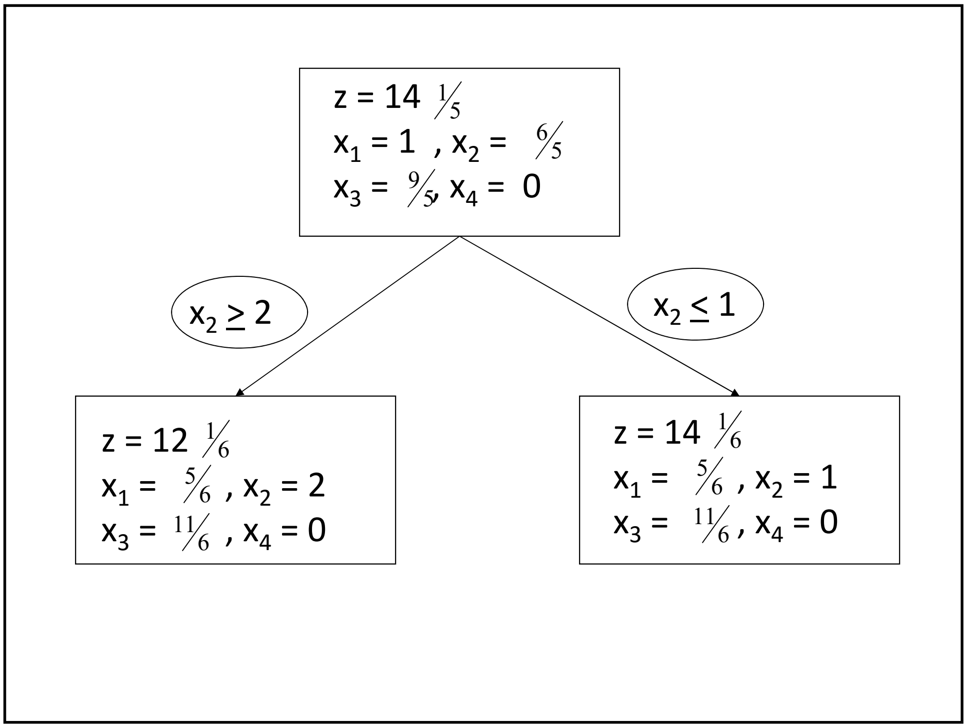
$$\begin{aligned} x_1 \quad\quad + 5x_3 \quad\quad &\leq 10 \\ x_1 + x_2 - x_3 \quad\quad &\leq 1 \\ 6x_1 - 5x_2 \quad\quad\quad &\leq 0 \\ -x_1 \quad\quad + 2x_3 - 2x_4 &\leq 3 \end{aligned}$$

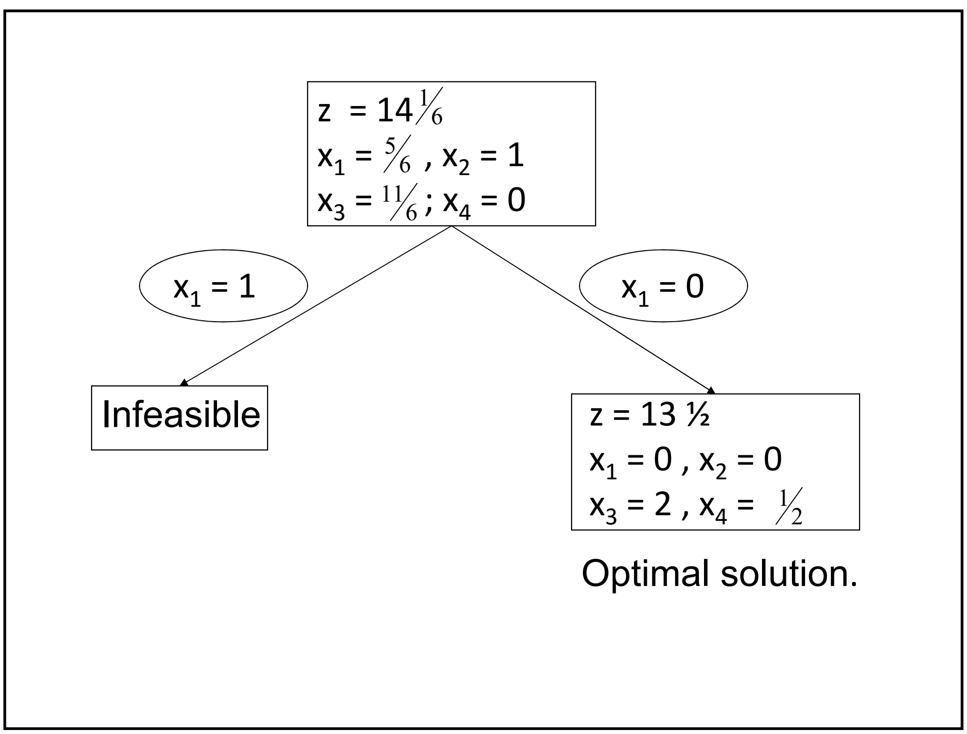
$x_j \geq 0$, for j = 1,2,3,4

x_j integer for j = 1,2,3

Solution tree after first iteration:




$z = 14\frac{1}{5}$
$x_1 = 1$, $x_2 = \frac{6}{5}$
$x_3 = \frac{9}{5}$, $x_4 = 0$
$x_2 \geq 2$
$x_2 \leq 1$
$z = 12\frac{1}{6}$
$x_1 = \frac{5}{6}$, $x_2 = 2$
$x_3 = \frac{11}{6}$, $x_4 = 0$
$z = 14\frac{1}{6}$
$x_1 = \frac{5}{6}$, $x_2 = 1$
$x_3 = \frac{11}{6}$, $x_4 = 0$


$z = 14\frac{1}{6}$
$x_1 = \frac{5}{6}$, $x_2 = 1$
$x_3 = \frac{11}{6}$; $x_4 = 0$
$x_1 = 1$
$x_1 = 0$
Infeasible
$z = 13\frac{1}{2}$
$x_1 = 0$, $x_2 = 0$
$x_3 = 2$, $x_4 = \frac{1}{2}$
Optimal solution.

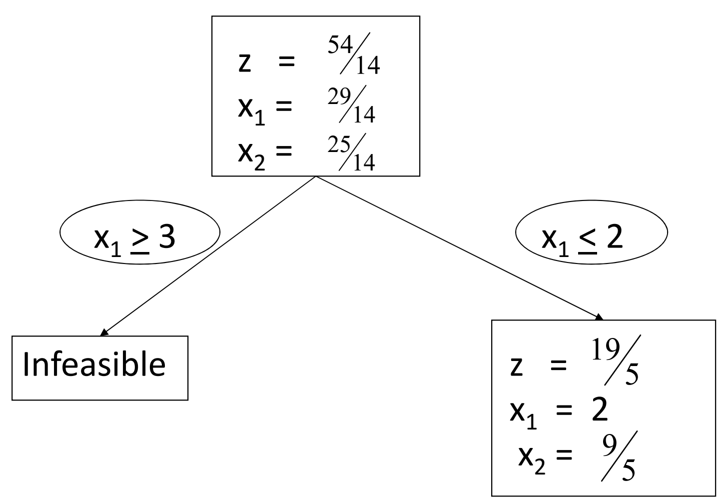
HOMEWORK 15

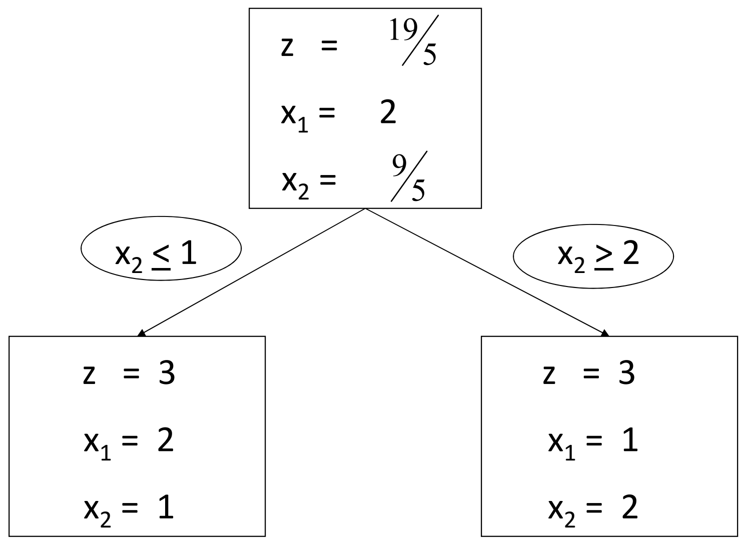
Solve the following integer programming problem.

$$\text{Maximize } z = x_1 + x_2$$

s.t.

$$x_1 + 5x_2 \leq 11$$
$$3x_1 + x_2 \leq 8$$
$$x_1, x_2 \geq 0$$
$$x_1 \text{ integer.}$$



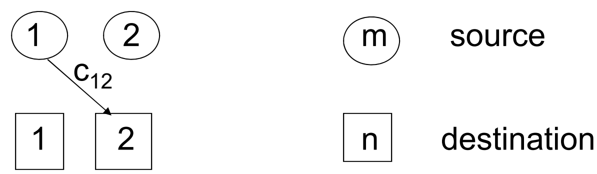


BINARY IPP

Max $z = 3x_1 + x_2$
s.t.
$4x_1 + 3x_2 \leq 10$
$x_1, x_2 \geq 0$
$x_1, x_2 \in \{0,1\}$

To solve, simply add
$x_1 \leq 1$
$x_2 \leq 1$
x_1, x_2 integer,
eliminate the binary constraint, and solve as a pure IPP.

TRANSPORTATION PROBLEM



The constant c_{ij} is the cost per unit from source i to destination j, while x_{ij} is the amount (integer number of units) shipped from source i to destination j.

The aim of the model is to minimize the shipping cost, i.e.,

$$\text{Min } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

s.t.

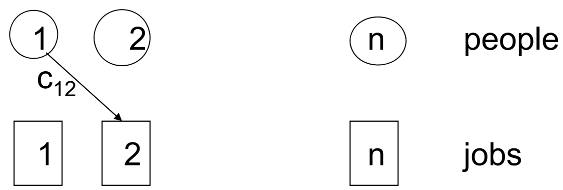
$$\sum_{j=1}^{n} x_{ij} = s_i \,, \; i = 1,\ldots,m$$

$$\sum_{i=1}^{m} x_{ij} = d_j \,, \; j = 1,\ldots,n$$

x_{ij} non-negative integer.

The transportation problem can be solved by the simplex algorithm.

ASSIGNMENT MODEL



Here, c_{ij}=cost for assigning person i to do the job j.

$$\text{Min } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

s.t.

$$\sum_{j=1}^{n} x_{ij} = 1, \ i = 1,\ldots,n$$

$$\sum_{i=1}^{n} x_{ij} = 1, \ j = 1,\ldots,n$$

$$x_{ij} \in \{0,1\}.$$

The assignment model is a binary integer programming problem that can be solved as a transportation model.

PRACTICE TEST 1

1. Apply the simplex method to the following linear programming problem.

$$\text{Maximize } z = 6x_1 - 3x_2$$

s.t.

$$x_1 + 2x_2 \leq 8$$
$$x_1 - x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

Answer: $z = 28$, $x_1 = 16/3$, $x_2 = 4/3$.

2. Apply the simplex method to the following linear programming problem.

$$\text{Minimize } z = -5x_1 + 6x_2$$

s.t.

$$-2x_1 + x_2 \geq 8$$
$$5x_1 + 2x_2 = 20$$
$$x_1, x_2 \geq 0.$$

Answer: $z = 460/9$, $x_1 = 4/9$, $x_2 = 80/9$.

3. The Crazy Nut Company wishes to market two special nut mixes during the holiday season. Mix 1 contains 2/3 pound of peanuts and 1/3 pound of cashews; mix 2 contains 3/5 pound of peanuts, 1/4 pound of cashews, and 3/20 pound of almonds. Mix 1 sells for $1.49 per pound; mix 2 sells for $1.69 per pound. The data pertinent to the raw ingredients appear in the table. Assuming that Crazy Nut can sell all cans of either mix that it produces, formulate an LP model to determine how much of mixes 1 and 2 the company should produce.

Ingredient	Amount Available (lb)	Cost per lb
Peanuts	30,000	$.35
Cashews	12,000	$.50
Almonds	10,000	$.70

Answer:

x_1 = number # mix 1

x_2 = number # mix 2

$$\text{Max } z = 1.09x_1 + 1.25x_2$$

s.t.

$$2/3x_1 + 3/5x_2 \leq 30000$$

$$1/3x_1 + 1/4x_2 \leq 12000$$

$$3/20x_2 \leq 10000$$

$$x_1, x_2 \geq 0.$$

4. Consider the linear programming problem

$\max z = 4x_1 + x_2$

s.t.

$2x_1 + 3x_2 \leq 6$

$x_1, \; x_2 \geq 0.$

The optimal tableau is given below, where x_3 is the slack variable added to the constraint.

BV	z	x_1	x_2	x_3	RHS
Z	1	0	5	2	12
x_1	0	1	3/2	1/2	3

Add the constraint $x_2 \geq 1$ to the original problem and solve it beginning with the above tableau.

Answer: $z = 7, \; x_1 = 3/2, x_2 = 1.$

5. Solve the following integer programming problem.

Maximize $z = x_1 + x_2$

s.t.

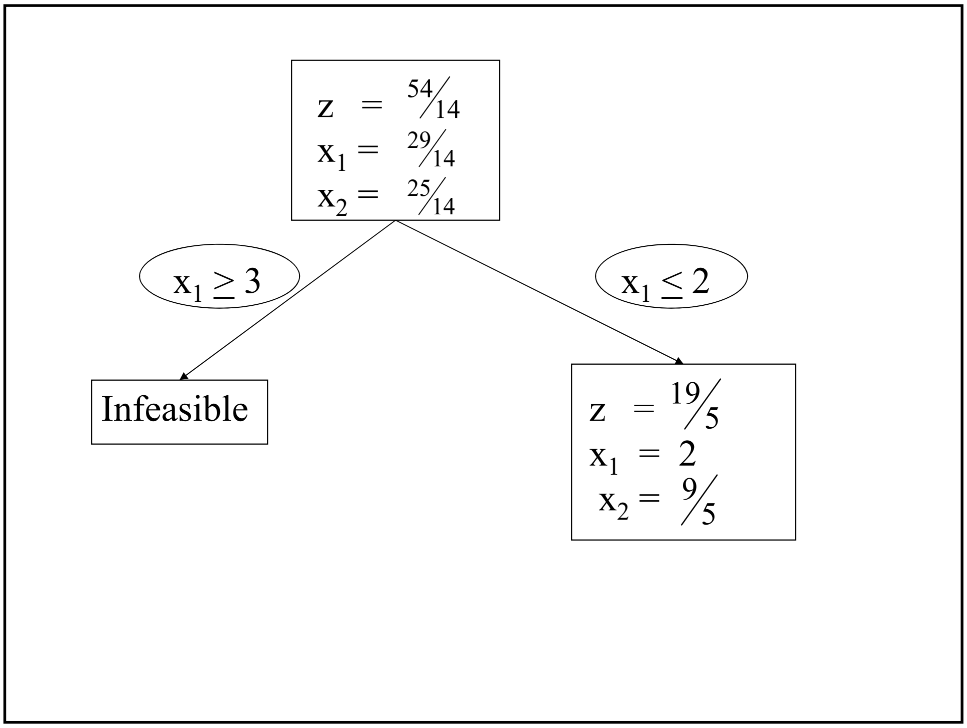
$x_1 + 5x_2 \leq 11$

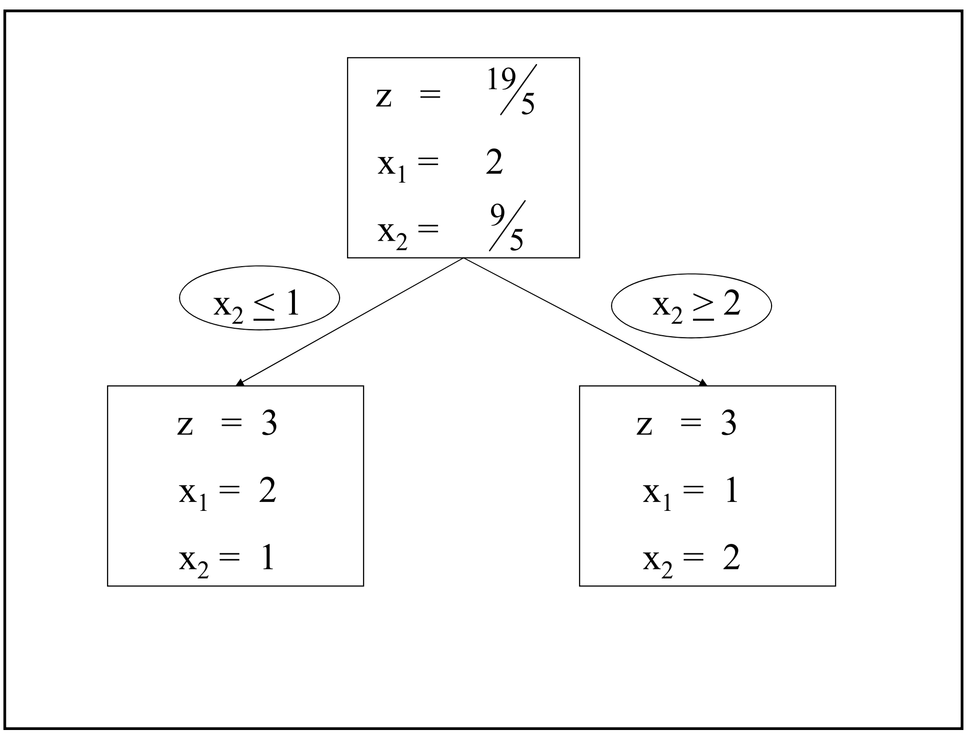
$3x_1 + x_2 \leq 8$

$x_1, x_2 \geq 0$

x_1, x_2 integer.

Solution on next page:


$z = 54/14$
$x_1 = 29/14$
$x_2 = 25/14$
$x_1 \geq 3$
$x_1 \leq 2$
Infeasible
$z = 19/5$
$x_1 = 2$
$x_2 = 9/5$


$z = 19/5$
$x_1 = 2$
$x_2 = 9/5$
$x_2 \leq 1$
$x_2 \geq 2$
$z = 3$
$x_1 = 2$
$x_2 = 1$
$z = 3$
$x_1 = 1$
$x_2 = 2$

REVIEW FOR QUIZ 1

NONLINEAR PROGRAMMING

$$\max\ f(x_1, x_2, x_3) = x_1^2 + x_1 x_2 + x_1 x_3$$

$s.t.$

$$\frac{x_1}{x_2} + x_3^2 \le 100$$

$$x_1 + 2x_2 x_3 \ge 250$$

$$x_1, x_2, x_3 \ge 0$$

Applications of NLP

- Data networks – routing
- Production planning
- Resource allocation
- Modeling human or organizational behavior

Example 1

Given a fixed area of cardboard $\mathrm{A\ unit}^2$, formulate a nonlinear program to find the dimension of a six-sided rectangular box with maximum volume.

$$\begin{aligned} &max\ xyz \\ &s.t.\quad 2xy + 2xz + 2yz = A \\ &\qquad\quad x, y, z \geq 0 \end{aligned}$$

Example 2

Consider the problem of determining locations for two new high schools in a set of P subdivisions N_j. Let w_{1j} be the number of students going to school A and w_{2j} be the number of students going to school B from subdivision N_j . Assume that the student capacity of school A is c_1 and the capacity of school B is c_2 and that the total number of students in each subdivision is r_j . We would like to minimize the total distance traveled by all the students given that they may attend either school A or B. Construct a nonlinear program to determine the locations (a, b) and (c, d) of high schools A and B, respectively assuming the location of each subdivision N_i is modeled as a single point denoted (x_i, y_i).

Answer

$$minimize \sum_{j=1}^{P} w_{1j}\left(\left(a-x_j\right)^2+\left(b-y_j\right)^2\right)^{\frac{1}{2}}$$

$$+ w_{2j}\left(\left(c-x_j\right)^2+\left(d-y_j\right)^2\right)^{\frac{1}{2}}$$

$$s.t.$$

$$\sum_j w_{ij} \le c_i\,, i = 1, 2;\; j = 1, \dots, P$$

$$w_{1j} + w_{2j} = r_j, \qquad j = 1, \dots, P$$

Notation

- $f : R^1 \rightarrow R^1$ means $f(x) = y$
- $f : R^n \rightarrow R^1$ means $f(x_1, \ldots x_n) = y$
- In general we want to

$$\max f(x_1, \ldots x_n)$$
$$s.t.$$
$$(x_1, \ldots x_n) \in A \subset R^n$$

where A is the feasible region.

Classical Optimization in R^1

- Let $f : R^1 \rightarrow R^1$. We first wish to

(P1) maximize $f(x)$

s.t.

$x \in R^1$

where f is differentiable. Here, there are no constraints and the feasible region is R^1.

Definitions

1. The point x* is a local maximum if $f(x) \leq f(x^*)$ for all x in some neighborhood of x*.
2. The point x* is a global maximum if $f(x) \leq f(x^*)$ for all x.
3. The term maximum means global maximum and is a point in the domain.

SOME TERMS IN LOGIC

- Let p and q be propositions - statements that can be judged true or false.
- <u>Necessary condition</u>
  Consider the proposition: "If p, then q."
  Then q is said to be a necessary condition for p.

- Sufficient condition

  Consider the proposition: "If p, then q."

  Then p is said to be a sufficient condition for q.

- Contrapositive

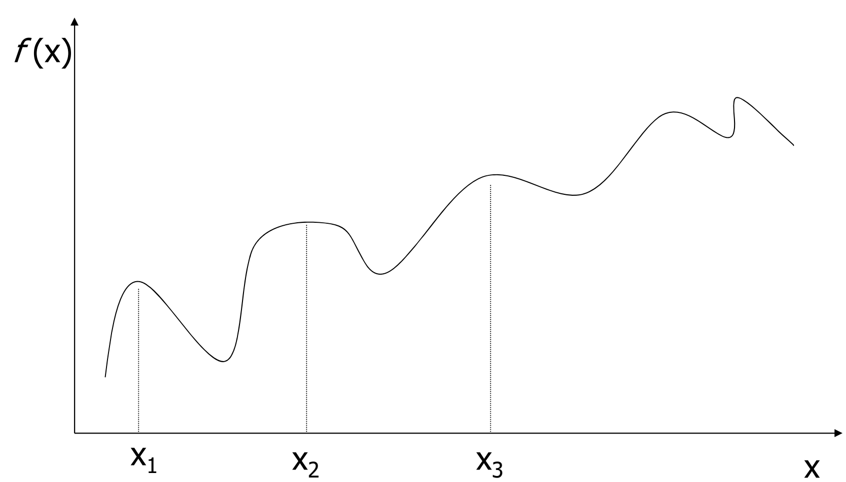
  Consider the following two propostions.

  (a) "If p, then q."

  (b) "If not q, then not p."

  Then (b) is called the contrapositive of (a).

Definitions



Results

- Necessary condition from calculus
  - If x* solves P1, then $f'(x^*)=0$.
- Sufficient condition 1
  - Let x* be an stationary point (with $f'(x^*)=0$). Then if $f''(x^*) < 0$, x* is a local maximum.
- Sufficient condition 2
  - If the order of the first non-vanishing derivative is even and the value at x* is < 0, then x* is a local max.

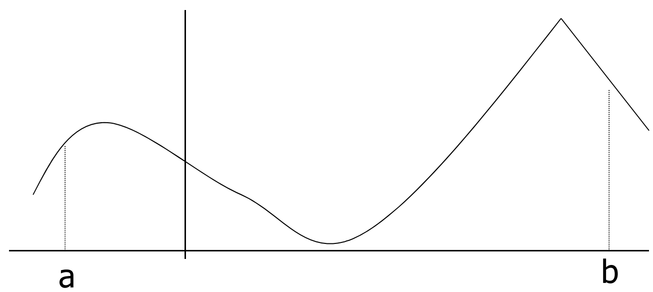
Critical Points

Any extremum of $f(x)$ over [a,b] must be one of the following critical points:

1. A stationary point in [a,b] where $f'(x)=0$
2. a,b
3. Points in [a,b] for which $f'(x)$ is not defined.

Critical Point Problem

- Optimize *f* (x)
  s.t.
  x∈[a,b]



Example

Optimize *f* (x) = (x-2)^3 + |x|
s.t.
x ∈ [-1,3]

To find the stationary points we write

$$f(x) = \begin{cases} (x-2)^3 + x, x \geq 0 \\ (x-2)^3 - x, x \leq 0 \end{cases}$$

It follows that

$$f'(x) = \begin{cases} 3(x-2)^2 + 1, \; x > 0 \\ 3(x-2)^2 - 1, \; x < 0. \end{cases}$$

Thus

$$3x^2\text{-}12x\text{+}13\text{=}0, \; x > 0$$
$$3x^2\text{-}12x\text{+}11\text{=}0, \; x < 0.$$

So

$$x = +2 \pm \frac{\sqrt{-3}}{3}, \; x > 0$$
$$x = +2 \pm \frac{\sqrt{3}}{3} \cong 1.433, 2.577, \; x < 0.$$

In both cases, the solutions are not negative and hence there are no stationary points for $x < 0$.
The critical points are thus

1. $\varnothing$
2. $x = -1,3$
3. $x = 0.$

To obtain the optima, we evaluate the objective function at the critical points.

	x	$f(x)$	
min	-1	-26	←
	0	-8	
max	3	4	←

HOMEWORK 16

Minimize $f(x) = |x+3| + x^3$

s.t.

$x \in [-2,6]$

.

Unconstrained Problems in R^n

- Let $f : R^n \rightarrow R^1$
- Let $\mathbf{x} \in R^n$
- $$Q(x) = x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$
- Q(x) is called a quadratic form, where the matrix **A** = $[a_{ij}]$ is symmetric.

Q(x) is called:

1. Positive definite if $Q(x)>0, \forall\ x \neq 0$
2. Positive semidefinite if $Q(x) \geq 0, \forall\ x$ and $\exists x \neq 0$ such that $Q(x)=0$
3. Negative definite if $Q(x)<0, \forall\ x \neq 0$
4. Negative semidefinite if $Q(x) \leq 0, \forall\ x$ and $\exists x \neq 0$ such that $Q(x)=0$
5. Indefinite otherwise

Positive/Negative Definite

- $Q(x) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ is positive definite if the values of the leading principal minor determinants of **A** are all positive. **A** is called positive definite in this case.
- $Q(x) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ is negative definite if the value of the k^{th} leading principal minor determinant has the sign $[-1]^k$, k=1,2,…,n. **A** is called negative definite in this case.

Hessian Matrices

- In particular, we are interested in determining whether the Hessian matrix **H(x)** of a function *f* at point x = (x_1,…x_n) is positive definite, etc.
- The Hessian is defined as the n x n matrix

$$H(x) = \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right].$$

Example

- Show that the Hessian matrix of the function f(x) below is negative definite for all x.

$$f(x_1, x_2) = x_1 + 3x_2 - 4x_1x_2 - 2x_1^2 - 3x_2^2 - 7$$

- H = $\begin{bmatrix} -4 & -4 \\ -4 & -6 \end{bmatrix}$ for all x

- Then the principal minor determinants are
  -4 and 8

- Thus the Hessian of f(x) is negative definite for all x.

Necessary and Sufficient Optimality Conditions

- Necessary condition
  - A necessary condition for X_0 to be an extreme point of f(X) is that $\nabla f(X_0) = 0$
- Sufficient condition
  - A sufficient condition for a stationary point X_0 to be an extremum is for the Hessian matrix **H** evaluated at X_0 to be
    - Positive definite when X_0 is a local minimum point
    - Negative definite when X_0 is a local maximum point.

Example

- Consider the function

$$f(x_1, x_2, x_3) = x_1 + 2x_3 + x_2x_3 - x_1^2 - x_2^2 - x_3^2.$$

- The necessary condition $\nabla f(X_0) = 0$ gives

  $X_0 = (1/2, 2/3, 4/3)$.

Example

- To establish sufficiency compute

$$H|_{X0} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

- The principle minor determinants have the values
  - – 2,4, and – 6.
- Thus $H|_{X0}$ is a negative definite and X_0 = (1/2, 2/3, 4/3) represents a local maximum point.

Convex and Concave Functions

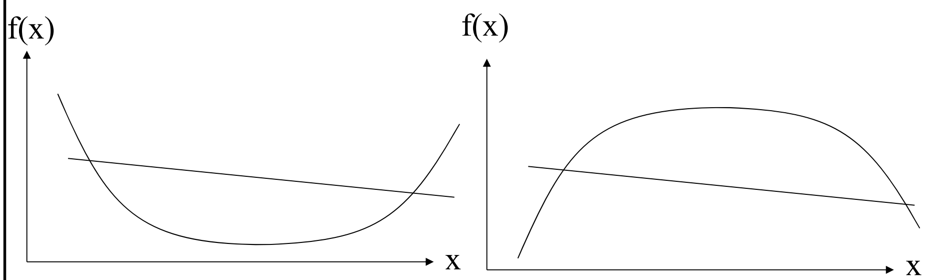
- A function of $f : R^n \rightarrow R^1$ is convex on a convex set S if for any two points $x_1, x_2 \in$ S and any $\lambda \in [0,1]$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

- A function of $f : R^n \rightarrow R^1$ is concave on a convex set S if for any two points $x_1, x_2 \in$ S and any $\lambda \in [0,1]$

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$$

Convex and Concave Functions



Convex and Concave Functions

- A function *f* is convex (concave) on a convex set S if and only if its Hessian matrix **H(x)** is positive (negative) semidefinite for all $x \in S$.
- A convex function achieves its global minimum at a stationary point.
- A concave function achieves its global maximum at a stationary point.

- For example we know that the function f below is concave on R^2 because we have already shown that its Hessian matrix is negative semidefinite for all (x_1,x_2).

$$f(x_1,x_2)=x_1+3x_2-4x_1x_2-2x_1^2-3x_2^2-7$$

HOMEWORK 17

Check if the following function is convex or concave:

$$f(x_1,x_2) = x_1 + 3x_1x_2 + x_2^2 - 7.$$

Constrained Problems in R^n

(P) $\max f(x_1,\ldots,x_n)$

$s.t.$

$$g_i(x_1,\ldots,x_n) \geq 0, i = 1,\ldots,m$$

$$h_k(x_1,\ldots,x_n) = 0, i = 1,\ldots,p$$

Checking for Short Cuts

- Check to see if you can find a solution to P by ignoring some or all of the constraints. If this solution also satisfies the neglected constraints, then it is optimal to P.

- Check to see if you can reduce the number of variables in a problem by solving some of the equality constraints for variables and substituting into the objective function and remaining constraints.

EXAMPLE 1

$$\text{minimize } (x_1 - 1)^2 + (x_2 - 3)^2$$

s.t.

$$x_1^{\,3} + x_2^{\,.25} \le 4$$

$$|\sin(x_1 x_2 \pi)| \le \frac{\sqrt{3}}{2}.$$

By neglecting both constraints,the answer by inspection is $x_1^* = 1,\ x_2^* = 3$.

EXAMPLE 2

$$\text{maximize } x^{\frac{1}{3}}$$

s.t.

$$|x| \le 1$$

$$\text{Sin}^{-1}(x) \ge 1.5\ .$$

By neglecting the second constraint, the answer by inspection is $x^* = 1$.

EXAMPLE 3

(I) maximize $x_1(1-x_2)$

s.t.

$$x_1 + x_2 = 1$$

$$x_1{}^2 + x_2{}^2 \le 1.$$

From the first constraint, $x_2 = 1 - x_1$.

Substitution gives the problem

(II) maximize $x_1{}^2$

s.t.

$$x_1{}^2 + (1-x_1)^2 \le 1.$$

The constraint to (II) reduces to

$$2x_1^2 - 2x_1 \le 0$$

or

$$x_1(x_1 - 1) \le 0.$$

By taking cases this gives

$$0 \le x_1 \le 1.$$

The answer to (II) is thus $x_1^* = 1$ by inspection and to (I) is $x_1^* = 1, x_2^* = 0.$

HOMEWORK 18

$$\text{minimize } x_1^2 + x_2^2$$
$$\text{s.t.}$$
$$x_1 + x_2 = 1$$
$$\left(x_1^{1/2}\right)\cos(\pi x_2) \le 1/2$$
$$x_1, x_2 \ge 0.$$

TO FIND A GLOBAL MAXIMUM FOR PROBLEM P IN PRACTICE:

- Find all interior points $(x_1,\ldots,x_n)$ for which the partials are 0 (stationary points).
- If f is not differentiable everywhere, include also points where the partials $\frac{\partial f}{\partial x_i}$ do not exist together with stationary points to give critical points.
- Find all boundary points $(x_1,\ldots,x_n)$ satisfying

  $\mathbf{d} \cdot \nabla f(x_1,\ldots,x_n) \ge 0$ for all feasible directions **d**.
- Compare values at all these points and choose the largest value. The points giving this largest value are the global maxima if a maximum exists.

The following conditions provide a method for doing this.

FRITZ JOHN NECESSARY CONDITIONS FOR PROBLEM P

Suppose that $f, g_1, \cdots, g_m,$ and $h_1, \cdots, h_p$ are all differentiable at the maximum $(x_1^*, \cdots, x_n^*)$ for P. Then there exist real numbers (Lagrange multipliers) $\alpha, \mu_1, \cdots, \mu_m,$ and $\lambda_1, \cdots, \lambda_p$ such that $(x_1^*, \cdots, x_n^*)$ and these constants satisfy the following conditions:

Fritz John Necessary Conditions

1. $\left. \alpha \dfrac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \mu_i \dfrac{\partial g_i}{\partial x_j} + \sum_{k=1}^{p} \lambda_k \dfrac{\partial h_k}{\partial x_j} \right|_{(x_1^*, \ldots x_n^*)} = 0, \; j = 1, ..n$
2. $\mu_i g_i(x_1^*, \ldots, x_n^*) = 0, i = 1, \ldots, m$
3. $\alpha, \mu_i \geq 0, i = 1, \ldots, m$
4. $\alpha, \mu_1, \ldots, \mu_m, \lambda_1, \ldots, \lambda_p$ are not all 0
5. $g_i(x_1^*, \ldots x_n^*) \geq 0, i = 1, \ldots, m$

   $h_k(x_1^*, \ldots x_n^*) = 0, i = 1, \ldots, p$

Karush-Kuhn-Tucker Necessary Conditions

- Very frequently the Fritz John conditions can be simplified somewhat because the constraints are “nice” at $(x_1^*,\ldots x_n^*)$. In that case α=1 and condition 4 is automatically satisfied.
- For our purposes, we take α=1 only when all constraints are linear (or Nota Bene applies) although the optimum usually occurs when α=1. Otherwise the necessary conditions do not involve the objective function.

Procedure

- Solve the Fritz John conditions to get the candidates for the maximum P.
- If the solution exists, it must be among the candidates.
- Evaluate all candidates.
- The best one is maximum **<u>if</u>** a maximum exists.

Nota Bene

- If you're lucky, then
  - The objective function *f* and all of the inequality constraints *g* are concave.
  - The equality constraints *h* are linear.
- In this case, the KKT conditions are both necessary and sufficient.
- $\alpha = 1$.
- No need to check all candidate solutions.

Weierstrass (Extreme Value) Theorem

- A continuous function on a closed bounded region achieves both its maximum and minimum on the region.
- It follows that if all the functions in P are continuous and the feasible region is bounded, then P has a solution.

HOMEWORK 19-23

Five nonlinear programming problems solved by Lingo, Matlab, or some other standard software are due on March 20 at the beginning of class. These problems must be formulated by the student, so each student will have different problems. A trial version of Lingo can be obtained at www.lindo.com. A copy of your problems and some form of computer printout of the solution are required.

EXAMPLE 1

$$\min x_1^2 + x_2^2$$

$$s.t.$$

$$x_1 + 2x_2 \le 3$$

$$2x_1 + x_2 \le 3$$

Example 1 in Standard Form:

$$\max\ f(x_1, x_2) = -x_1^{\ 2} - x_2^{\ 2}$$

$$s.t.$$

$$g_1(x_1, x_2) = 3 - x_1 - 2x_2 \geq 0$$

$$g_2(x_1, x_2) = 3 - 2x_1 - x_2 \geq 0$$

Example 1 - Questions

- Are all constraints linear?
  - Yes, so α=1
- Is the feasible region bounded?
  - No
  - We must worry about existence.

Example 1-Fritz John Conditions

$$(1)\begin{cases}-2x_1-\mu_1-2\mu_2=0\\-2x_2-2\mu_1-\mu_2=0\end{cases}$$

$$(2)\begin{cases}\mu_1(x_1+2x_2-3)=0\\\mu_2(2x_1+x_2-3)=0\end{cases}$$

$$(3)\mu_1,\mu_2\geq 0$$

$$(4) okay$$

$$(5)\begin{cases}3-x_1-2x_2\geq 0\\3-2x_1-x_2\geq 0\end{cases}$$

Example 1 – Cases from condition (3)

- Case (a) μ_1=0, μ_2=0

$$(1)\Rightarrow x_1=x_2=0$$

- Case (b):μ_1=0, μ_2>0

$$(1)\Rightarrow\begin{cases}-2x_1-2\mu_2=0\Rightarrow x_1=-\mu_2\\-2x_2-\mu_2=0\Rightarrow x_2=-\mu_2/2\\\therefore x_1=2x_2\end{cases}$$

$$(2)\Rightarrow 2x_1+x_2=3\Rightarrow x_1=6/5, x_2=3/5$$

$$(5) okay$$

Example 1 – Cases from condition (3)

- Case (c) $\mu_1>0$, $\mu_2=0$

$$(1) \Rightarrow \begin{cases} -2x_1 - 2\mu_1 = 0 \Rightarrow x_1 = -\mu_1/2 \\ -2x_2 - 2\mu_1 = 0 \Rightarrow x_2 = -\mu_1 \\ \therefore x_2 = 2x_1 \end{cases}$$

$$(2) \Rightarrow x_1 + 2x_2 = 3 \Rightarrow x_1 = 3/5, x_2 = 6/5$$

$$(5) okay$$

Example 1 – Cases from condition (3)

- Case (d) $\mu_1>0$, $\mu_2>0$

$$(2) \Rightarrow \begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 + x_2 = 3 \\ \therefore x_1 = x_2 = 1 \end{cases}$$

$$(5) okay$$

Example 1 – Evaluation of candidates

	(x_1,x_2)	$f(x_1,x_2)$
max	(0, 0)	0 ←
	(6/5, 3/5)	-45/25
	(3/5, 6/5)	-45/25
	(1, 1)	-2

If there is a max, it occurs at the point (0,0)

But wait ….

- We didn't ask a most important question - is there a short cut or easier way ?

$$\min x_1^{\ 2} + x_2^{\ 2}$$

$$s.t.$$

$$x_1 + 2x_2 \leq 3$$

$$2x_1 + x_2 \leq 3$$

Questions we should ask:

- Is there an easier way?
  - Yes
  - Then do it and save yourself lots of time.
- Are all constraints linear?
  - Yes, so α=1'
- Is the feasible region bounded?
  - No
  - We must worry about existence.

EXAMPLE 2

$$\min \; x_1 + 3x_2$$

$$s.t.$$

$$2x_1^2 + 5x_2^{\,2} = 230$$

Put in standard form.

$$\max\ f(x_1, x_2) = -x_1 - 3x_2$$

$$s.t.$$

$$h_1(x_1, x_2) = 2x_1^2 + 5x_2^{\ 2} - 230 = 0$$

Questions we should ask:

- Is there an easier way?
  - No
- Are all constraints linear?
  - No
- Is the feasible region bounded?
  - Yes
  - The best candidate is the solution.

Fritz John Conditions

$$(1)\begin{cases} -\alpha + 4x_1\lambda_1 = 0 \\ -3\alpha + 10x_2\lambda_1 = 0 \end{cases}$$

$(2) NA$

$(3) \alpha \geq 0$

$(4) \alpha, \lambda_1$ not both zero

$(5) 2x_1^2 + 5x_2^2 - 230 = 0$

Case I: $\alpha = 0$

$(5) \Rightarrow x_1$ and x_2 not both 0.

$(1) \Rightarrow 4x_1\lambda_1 = 0 = 10x_2\lambda_1.$

$\therefore\ \lambda_1 = 0 \otimes (4).$

Case II : $\alpha = 1$

$$(1) \Rightarrow \begin{cases} 4\lambda_1 x_1 = 1 \\ 10\lambda_1 x_2 = 3 \end{cases} \Rightarrow \lambda_1 \neq 0.$$

$$\therefore x_2 = \tfrac{6}{5} x_1.$$

$$(5) \Rightarrow 2x_1^2 + 5\left(\tfrac{6}{5}\,x_1\right)^2 = 230.$$

$$\therefore (x_1, x_2) = (5,6),\ (-5,-6).$$

Obviously both satisfy (5).

	(x_1, x_2)	$f(x_1, x_2)$
	(5,6)	–23
max	(–5,–6)	23 ⟵

Therefore the minimum to the original problem is (–5,–6).

EXAMPLE 3

$$\text{maximize } x_1 + x_2$$

$$\text{s.t.}$$

$$x_1^2 + x_2^2 = 1$$

$$x_1, x_2 \geq 0.$$

We rewrite in the standard form of P.

$$\text{maximize } f(x_1, x_2) = x_1 + x_2$$

$$\text{s.t.}$$

$$g_1(x_1, x_2) = x_1 \geq 0$$

$$g_2(x_1, x_2) = x_2 \geq 0$$

$$h_1(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0.$$

Questions we should ask:

- Is there an easier way?
  - No
- Are all constraints linear?
  - No
- Is the feasible region bounded?
  - Yes
  - The best candidate is the solution.

Write the Fritz John Conditions.

$$(1)\begin{cases}\alpha + \mu_1 + 2x_1\lambda_1 = 0 \\ \alpha + \mu_2 + 2x_2\lambda_1 = 0\end{cases}$$

$$(2)\begin{cases}\mu_1 x_1 = 0 \\ \mu_2 x_2 = 0\end{cases}$$

$$(3)\alpha, \mu_1, \mu_2 \geq 0$$

$$(4)\alpha, \mu_1, \mu_2, \lambda_1 \text{ not all zero}$$

$$(5)\begin{cases}x_1 \geq 0 \\ x_2 \geq 0 \\ x_1^2 + x_2^2 - 1 = 0\end{cases}$$

Case I: $\alpha = 0$

subcase (a): $\mu_1 = 0, \mu_2 = 0$

$(4) \Rightarrow \lambda_1 \neq 0.$

$(1) \Rightarrow x_1 = x_2 = 0 \otimes (5).$

subcase (b): $\mu_1 = 0, \mu_2 > 0$

$(2) \Rightarrow x_2 = 0.$

$(1) \Rightarrow \mu_2 = 0 \ \otimes \mu_2 > 0.$

subcase (c): $\mu_1 > 0, \mu_2 = 0$

$(2) \Rightarrow x_1 = 0.$

$(1) \Rightarrow \mu_1 = 0 \ \otimes \mu_1 > 0.$

subcase (d): $\mu_1 > 0, \mu_2 > 0$

$(2) \Rightarrow x_1 = x_2 = 0 \otimes (5).$

Case II : $\alpha = 1$

subcase (a) : $\mu_1 = 0, \mu_2 = 0$

$$(1) \Rightarrow \begin{cases} 1 + 2\lambda_1 x_1 = 0 \\ 1 + 2\lambda_1 x_2 = 0. \end{cases}$$

$\therefore \lambda_1 \neq 0.$

Hence $x_1 = x_2 = -\frac{1}{2\lambda_1}$.

$(5) \Rightarrow x_1^2 + x_1^2 = 1.$

$\therefore \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$ is a candidate

but not $\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$.

subcase (b) : $\mu_1 = 0, \mu_2 > 0$

$(2) \Rightarrow x_2 = 0.$

$(5) \Rightarrow x_1 = \pm 1.$

$\therefore$ (1,0) but not (-1,0) is a candidate.

subcase (c) : $\mu_1 > 0, \mu_2 = 0$

$\therefore$ a candidate is (0,1) as before.

subcase (d) : $\mu_1 > 0, \mu_2 > 0$

$(2) \Rightarrow x_1 = x_2 = 0 \otimes (5)$ as before.

	(x_1, x_2)	$f(x_1, x_2)$
	$(1,0)$	1
	$(0,1)$	1
max	$\left(\sqrt{1/2}, \sqrt{1/2}\right)$	$\sqrt{2}$ ←

EXAMPLE 4

$$\text{maximize} \quad x_1 \cdot x_2$$

$$s.t.$$

$$x_1^2 + x_2^{\ 2} \le 4$$

$$2x_1 + x_2 = 2$$

Put in standard form.

$$\max \ f(x_1, x_2) = x_1 \cdot x_2$$

$$s.t.$$

$$g_1(x_1, x_2) = -x_1^2 - x_2^{\ 2} + 4 \geq 0$$

$$h_1(x_1, x_2) \ 2x_1 + x_2 - 2 = 0$$

Questions we should ask:

- Is there an easier way?
  - No
- Are all constraints linear?
  - No
- Is the feasible region bounded?
  - Yes
  - The best candidate is the solution.

Fritz John Conditions

$$(1)\begin{cases}\alpha x_2 - 2\mu_1 x_1 + 2\lambda_1 = 0\\ \alpha x_1 - 2\mu_1 x_2 + \lambda_1 = 0\end{cases}$$

$$(2)\left\{\mu_1(-x_1^2 - x_2^2 + 4) = 0\right.$$

$$(3)\ \alpha, \mu_1 \geq 0$$

$$(4)\ \alpha, \mu_1, \lambda_1 \text{ not all zero}$$

$$(5)\begin{cases}-x_1^2 - x_2^2 + 4 \geq 0\\ 2x_1 + x_2 - 2 = 0\end{cases}$$

Case I : $\alpha = 0$

subcase (a) : $\mu_1 = 0$

$(1) \Rightarrow \lambda_1 = 0 \ \otimes (4).$

subcase (b) : $\mu_1 > 0$

$(2) \Rightarrow x_1{}^2 + x_2{}^2 = 4$

$(5) \Rightarrow\ 2x_1 + x_2 = 2 \Rightarrow x_2 = 2 - 2x_1$

$\therefore x_1{}^2 + \left(2 - 2x_1\right)^2 = 4.$

So $\left(x_1, x_2\right) = \left(0, 2\right), \left(8/5, -6/5\right).$

Both these satisfy (5) and are candidates.

Case II : $\alpha = 1$

subcase (a) : $\mu_1 = 0$

$$(1) \Rightarrow \begin{cases} x_2 + 2\lambda_1 = 0 \\ x_1 + \lambda_1 = 0 \end{cases}$$

$$\therefore x_2 = 2x_1.$$

$$(5) \Rightarrow 2x_1 + x_2 = 2$$

$$\therefore x_1 = 1/2, x_2 = 1.$$

$\left(1/2, 1\right)$ satisfies (5).

subcase (b) : $\mu_1 > 0$

$$(2) \Rightarrow x_1^{\,2} + x_2^{\,2} = 4$$

$$(5) \Rightarrow 2x_1 + x_2 = 2$$

These have already been solved.

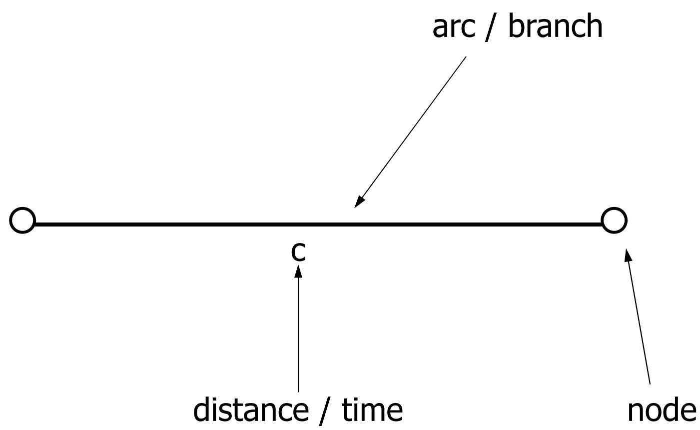
	(x_1, x_2)	$f(x_1, x_2)$
	$(0, 2)$	0
	$\left(\frac{8}{5}, -\frac{6}{5}\right)$	$-48/25$
max	$\left(\frac{1}{2}, 1\right)$	$1/2$ ←

HOMEWORK 24 - 27

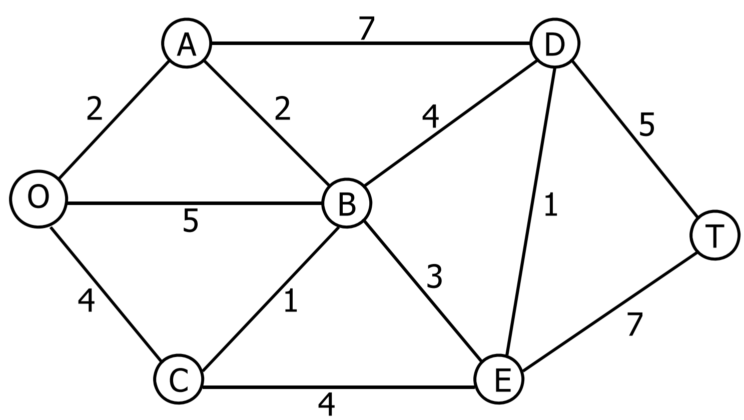
(24) maximize $2x_1 + 5x_2$
s.t.
$$2x_1^2 + 5x_2^2 = 13$$
(25) minimize $x_1 + 2x_2$
s.t.
$$x_1^2 + x_2^2 = 4$$
$$3x_1 + x_2 \le 4$$
(26) maximize $x_1 \cdot x_2$
s.t.
$$x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$
(27) maximize $x_1 \cdot x_2$
s.t.
$$x_1 + x_2 \le 4$$

REVIEW FOR QUIZ 2

NETWORK ANALYSIS



SHORTEST PATH



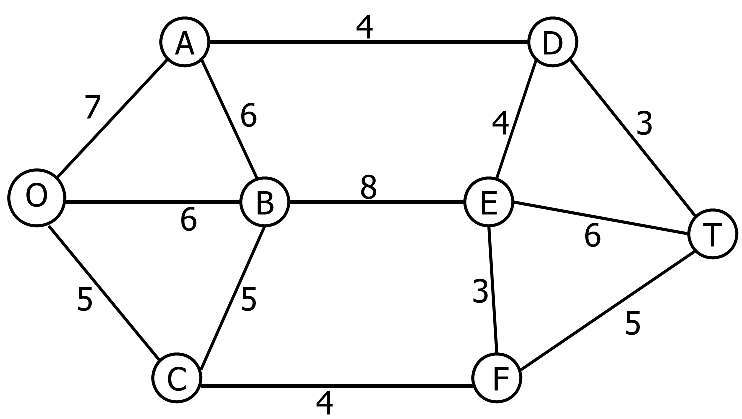
Note that the arcs / branches have no direction, ie., are undirected.

Seek a path from origin node O to terminal node T that minimizes the total distance, which is positive.

- Find all 1st closest nodes to O; then label them with the total distance to O and the preceding node in the path.
- Find the second closest nodes adjacent to the first closest nodes or origin.
- Proceeding, we eventually find T as the n^{th} closest node for some n.
- Working backwards gives the shortest paths.

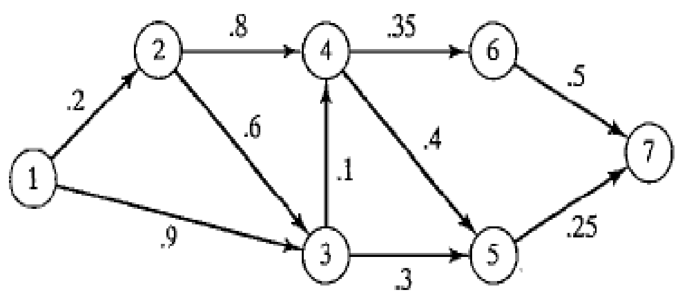
This is essentially Dijkstra's method. If arcs are directed, we can only do this in one direction.

HOMEWORK 28



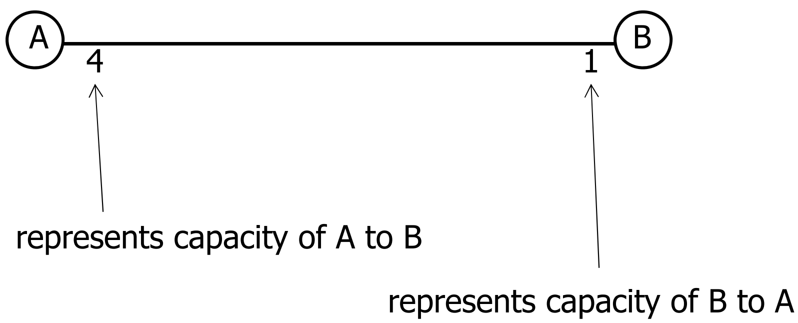
EXAMPLE

John drives to UTA every day. Having just completed a course in network analysis, John is able to determine the shortest route to work. Unfortunately, the selected route is heavily patrolled by police, and with all the fines paid for speeding, the shortest route may not be the best choice. John has thus decided to choose a route that maximizes the probability of not being stopped by police. The next figure shows the possible route between home and work, and the associated probabilities of not being stopped on each segment.

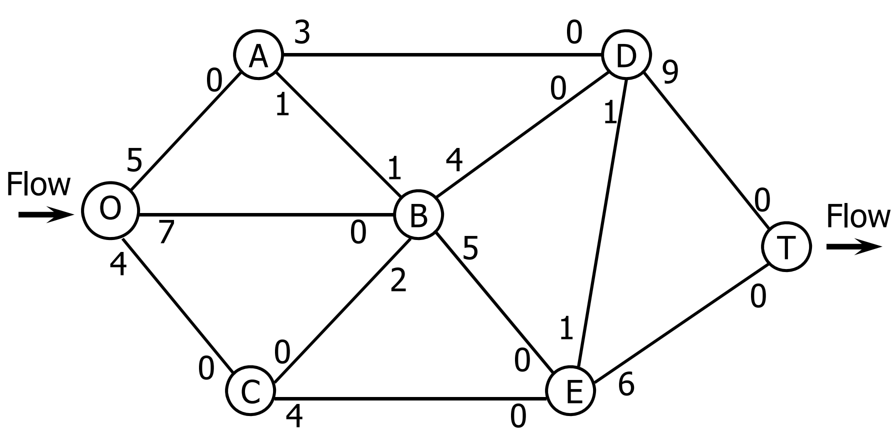


Shortest path: **1-3-5-7 = 0.0675**

FLOW IN NETWORKS



MAXIMAL FLOW



O = Source

T = Sink

PROCEDURE

1. Find a path with positive capacity from source to sink. Put that much flow in it.

2. Reduce the forward capacity by the flow and increase the backward capacity by the same amount to allow the possibility of undoing what you did.

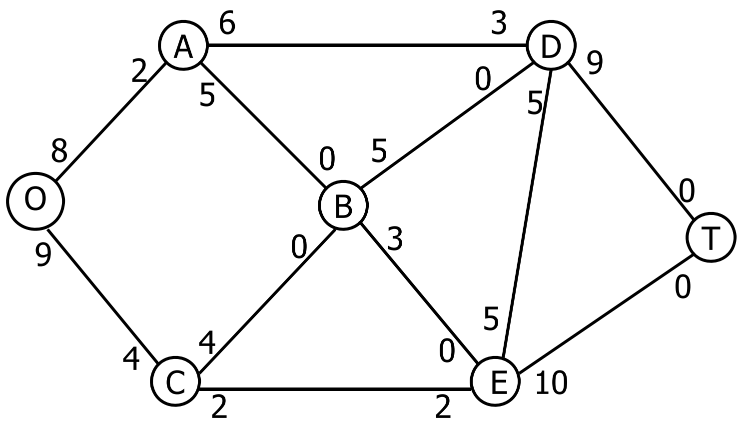
3. Repeat until this cannot be done. At that point you can find a cut with 0 remaining capacity from source side of cut to sink side.

4. For an example, see the link http://optlab-server.sce.carleton.ca/POAnimations2007/MaxFlow.html.

- **Definition**: A cut is a minimum set of branches whose breaking will separate the source from sink. Its cut value is the sum of the capacities of its branches from the source side of the cut to the sink side.

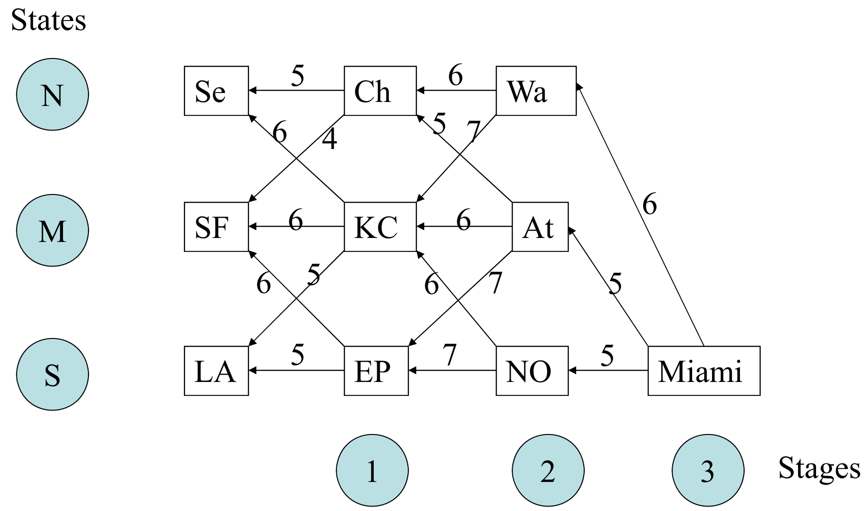
- **Max. Flow - Min. Cut Theorem**: The maximum flow through the network equals the minimum cut value, where the cut value of a cut is the sum of capacities from source to sink direction.

Homework 29



DYNAMIC PROGRAMMING

SURFING EXAMPLE



The goal of the above problem is to reach west coast from Miami in a minimum possible time.

Stage – where a decision is made, numbered here as the number of decisions left to make.

State - where you are in a stage.

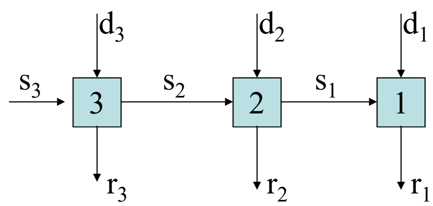
SOLUTION

stages: cities

states: north, middle, and south

decision variables: destination city at next stage

returns: time to reach next destination at next stage



Stage	State	time to best dstn.	rem. time
1	N	4 SF	4
	M	5 LA	5
	S	5 LA	5
2	N	6+4 ←	10
		7+5	
	M	5+4 ←	9
		6+5	
		7+5	

Stage	State	time to dstn.		rem. time
	S	6+5	←	11
		7+5		
3	S	6+10		
		5+9	←	14
		5+11		

ANSWER

Miami → At → Ch → SF

minimum time 5+5+4 = 14 hours

KEY POINTS OF DYNAMIC PROGRAMMING

1) Work backwards.
2) Break a problem into subproblems.
3) Bellman's principle of optimality: If a decision is to be made, make the best decision from here forward. Forget the past. Optimal subpolicies so obtained yield an optimal overall policy.

PROPERTIES OF SOLUTION PROCEDURE

1) A problem is divided into stages. A stage is where a decision is made.
2) DP transforms higher-dimensional problems into multiple lower-dimensional ones.
3) Each stage has a number of states associated with it. A state is "where you're at" in the stage.

4) One proceeds from a state in one stage to another state in the next stage. The optimal decision is found for each possible state of a stage, where optimal refers to future stages.

The "curse of dimensionality" refers to the fact that one could be required to solve large numbers of problem with large numbers of states.

5) Given the current stage, the optimal policy for the remaining stages is independent of the policy adopted in previous stages.

PROBLEM FORMULATION AND NOTATION

- d_n= decision variable at stage n
- s_n = state variable at stage n
- $r_n = r_n(s_n, d_n)$ = return at stage n in state s_n for decision d_n
- t_n= stage transformation $s_{n-1} = t_n(s_n, d_n)$ at stage n

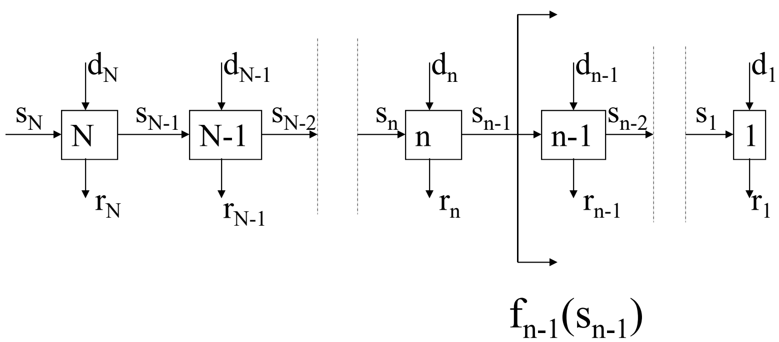
- $f_n(s_n)$ = optimal n stage return objective function that is only a function of state

$$f_n(s_n) = \underset{d_n}{\text{optimum}} \left[r_n(s_n, d_n) \otimes f_{n-1}(s_{n-1}) \right]$$

s.t

$$s_{n-1} = t_n(s_n, d_n), \quad n=2,3,\ldots\ldots,N.$$

$$f_1(s_1) = \underset{d_1}{\text{optimum}} \left[r_1(s_1, d_1) \right], \quad n=1.$$



STEPS IN SOLVING A PROBLEM

1) Define the mathematical optimization problem to be solved. This step may involve modeling a real-world situation.

2) Define the stages, the states, and the entities

$$d_n,\ s_n,\ r_n,\ t_n,\ f_n.$$

3) Give DP formulation to (1) by specifying the recursive equations of dynamic programming

$$f_n(s_n) = \underset{d_n}{\text{optimum}}\,[r_n(s_n, d_n) \otimes f_{n-1}(s_{n-1})]$$

s.t

$$s_{n-1} = t_n(s_n, d_n), \quad n = 2,3,\dots,N.$$

$$f_1(s_1) = \underset{d_1}{\text{optimum}}\,[r_1(s_1, d_1)], \quad n = 1.$$

4) Perform the computations.

Example 1: Allocation Problem

David Goodfellow has 3 children whose ages are 2, 3, 4. One day after work he decides to treat his children by buying them some candy. Unfortunately he has only 3¢, so he buys them 4 jellybeans. He desires, however, to utilize these 4 jellybeans to the fullest. He knows from experience that none of his children will eat more than 2 jellybeans. Moreover, being an observant father and knowing his children's characteristics, he estimates the units of pleasure that 0, 1, 2 jellybeans will give each child. The following table gives these estimates :

Units of Pleasure

jellybeans	Roosevelt	David Jr. (Davy)	Meridith
0	0	0	1
1	1	2	3
2	3	4	0

Suppose David, Sr. wishes to maximize the total pleasure that the jellybeans will give his children. Using dynamic programming, allocate the jellybeans optimally among the 3 children.

Solution

stages: children

states: # of jellybeans left

decision variables: how many jellybeans to give the child at stage n

returns: pleasure units (ahs!) at stage n

Stage 1

s_1	d_1	r_1	$f_1(s_1) = \max r_1$
4	0	0	
	1	1	
	2 *	3	3
3	0	0	
	1	1	
	2 *	3	3
2	0	0	
	1	1	
	2 *	3	3
1	0	0	
	1 *	1	1
0	0 *	0	0

Stage 2

s_2	d_2	r_2	$s_1 = s_2 - d_2$	$r_2 + f_1(s_1)$	$f_2(s_2)$
4	0	0	4	0 + 3 = 3	
	1	2	3	2 + 3 = 5	
	2 *	4	2	4 + 3 = 7	7
3	0	0	3	0 + 3 = 3	
	1 *	2	2	2 + 3 = 5	5
	2 *	4	1	4 + 1 = 5	5
2	0	0	2	0 + 3 = 3	
	1	2	1	2 + 1 = 3	
	2 *	4	0	4 + 0 = 4	4

Stage 3

s_3	d_3	r_3	$s_2 = s_3 - d_3$	$r_3 + f_2(s_2)$	$f_3(s_3)$
4	0 *	1	4	1 + 7 = 8	8
	1 *	3	3	3 + 5 = 8	8
	2	0	2	0 + 4 = 4	

Answer:

M	D	R
0	2	2
1	1	2
1	2	1

Example 2: Allocation Problem

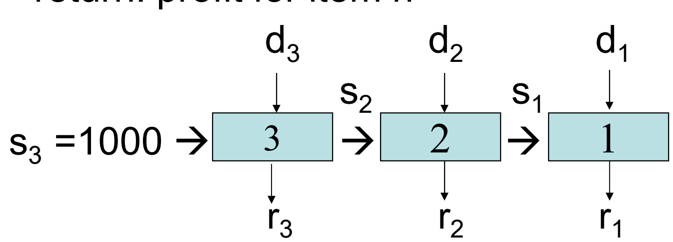
A small plane is being loaded with cargo consisting of 3 types of items. These items must be loaded in integer quantities , and the plane can carry only 1000 pounds of cargo. The profit realized from one unit of each type of item, as well as its weight per unit, is given below. Maximize the value of the cargo for the plane subject to the weight restriction.

Data

item i	Weight per unit	Profit per unit
A	250	$ 3000
B	300	$ 4000
C	400	$ 5000

Solution

- stages: items
- states: remaining weight available to allocate
- decision variables: units of item n
- return: profit for item n



Stage 1 - C

s_1	d_1*	$f_1(s_1)$
$0 \le s_1 < 400$	0	0
$400 \le s_1 < 800$	1	5000
$800 \le s_1 \le 1000$	2	10,000

Stage 2 - B

s_2	d_2	r_2	s_1	$r_2+f_1(s_1)$	$f_2(s_2)=\max[r_2+f_1(s_1)]$
1000	0	0	1000	0 + 10,000	
	1	4000	700	4000 + 5000	
	2 *	8000	400	8000 + 5000	13,000
	3	12,000	100	12000 + 0	
750	0	0	750	0 + 5000	
	1 *	4000	450	4000 + 5000	9000
	2	8000	150	8000 + 0	
500	0 *	0	500	0 + 5000	5000
	1	4000	200	4000 + 0	
250	0 *	0	250	0 + 0	0
0	0 *	0	0	0 + 0	0

Stage 3 - A

s_3	d_3	r_3	s_2	$r_3+f_2(s_2)$	$f_3(s_3)=\max [r_3+f_2(s_2)]$
1000	0 *	0	1000	0 + 13,000	13,000
	1	3000	750	3000 + 9000	
	2	6000	500	6000 + 5000	
	3	9000	250	9000 + 0	
	4	12,000	0	12,000 + 0	

Answer :

item	optimal #
A	0
B	2
C	1

HOMEWORK 30

A student has final examinations in 3 courses X,Y, Z, each of which is a 3 credit-hour course. He has 12 hours available for study period. He feels that it would be best to break the 12 hours up into 3 blocks of 4 hours each and to devote each 4-hour block to one particular course. His estimates of his grades based on various numbers of hours devoted to studying each course are as follows. Using dynamic programming, allocate his study time optimally.

Data

Number of hours

Course		0	4	8	12
	X	F	D	D	B
Course	Y	D	D	B	A
	Z	F	D	B	B

Example 3: Allocation Problem

Consider a system with 3 components in series of types A, B, and C respectively. If one component fails, the system fails. The reliability of the system (that is, the probability that all types of components work properly) can be improved by installing redundant components in parallel. Suppose that the unit cost and probability of failure of each type of component is given below.

Type	Failure Probability	Cost
A	0.6	$1
B	0.4	$2
C	0.5	$3

Assume that you have $10 to spend on components. This answer includes money for at least one of each type. Use dynamic programming to determine the optimum number of components of each type to buy so as to maximize the reliability of the system.

Answer

$$\underset{d_1,d_2,d_3}{\text{Maximize}} \quad [1-0.6^{d_1+1}][1-0.4^{d_2+1}][1-0.5^{d_3+1}]$$

s.t.

$$1d_1 + 2d_2 + 3d_3 \le 4$$

$$d_1, d_2, d_3 = 0, 1, 2, 3, \cdots$$

- stages: components
- states: s_i = $ left to spend at stage i
- decision variables: d_n = # redundancies of type n
- returns: r_n = prob. of success at stage n

$$r_n(d_n, s_n) = 1 - q_n^{d_n+1}$$

- stage transformations

$$s_{n-1} = s_n - c_n d_n$$

Stage 1: type A

s_1	d_1^*	$r_1 = (1-0.6^{d_1+1})$	$f_1(s_1) = \max(1-0.6^{d_1+1})$
4	4	1- 0.6^5 =0.92	0.92
3	3	1- 0.6^4 =0.87	0.87
2	2	1- 0.6^3 =0.78	0.78
1	1	1- 0.6^2 =0.64	0.64
0	0	1- 0.6^1 =0.40	0.40

Stage 2: type B

s_2	d_2	$r_2 = 1-0.4^{d_2+1}$	$s_1 = s_2 - 2d_2$	$r_2 \times f_1(s_1)$	$f_2(s_2) = \max[r_2 \times f_1(s_1)]$
4	0	0.6	4	0.60 $\times$ .92=0.55	
	1 *	0.84	2	0.84 $\times$ 0.78=0.65	0.65
	2	0.94	0	0.94 $\times$ 0.40=0.37	
1	0 *	0.6	1	0.6 $\times$ 0.64=0.38	0.38

Stage 3: type C

s_3	d_3	$r_3 = 1-0.5^{d_3+1}$	$s_2 = s_3 - 3d_3$	$r_3 \times f_2(s_2)$	$f_3(s_3) = \max[r_3 \times f_2(s_2)]$
4	0 *	0.5	4	0.5 $\times$ 0.65=0.325	0.325
	1	0.75	1	0.75 $\times$ 0.38=0.29	

Maximum reliability = 0.325

type	redundancies	total number
C	0	1
B	1	2
A	2	3

Example 4: Optimal Stopping Rule

An oral examination is designed as follows. There are 4 questions of which everyone is required to attempt the first question. If this first question is answered correctly, a student receives a numerical grade of 50 in the course. Otherwise, he receives a 0. For the remaining 3 questions , a student has the option of keeping his numerical grade from the previous question or attempting the next question for a higher grade. At each question, if the question is not answered correctly, the student receives a consolation grade.

The probabilities of correctly answering the three optional questions, the grade received for a correct response, and the consolation grade are given below. Formulate an optimal policy for taking this oral examination to maximize the expected numerical grade you receive.

Optional Question		Probability	Grade	Consolation Grade
α	(first)	0.6	65	40
β	(second)	0.5	80	55
γ	(third)	0.3	100	70

Answer

- stages: optional questions
- decision variable: stop or go
- returns: expected grade
- states: grade in hand

Stage 1: **Question γ**

Max{80, 0.3(100)+0.7(70) = 79}= 80   Stop

Stage 2: **Question** β

Max{65, 0.5(80)+0.5(55) = 67.5}= 67.5   Go

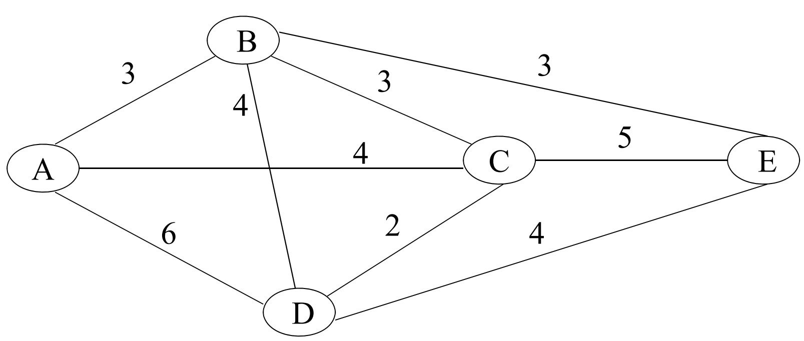
Stage 3: **Question** α

Max{50, 0.6(67.5)+0.4(40) = 56.5}= 56.5   Go

Question	Decision
α (first)	Go
β (second)	Go
γ (third)	Stop

Example 5: Traveling Salesman Problem

A businessman must travel to each of the following cities B, C, D, E, starting from A and ending in A. He can go through each city only once except that, of course, he ends in A after starting there. The "map" below indicates the possible routes that he could take, where the numbers represent distances. Find the optimal route that he should follow to minimize the total distance that he travels.



Solution

stages: cities (first three stops)

states: visited cities in order

decision variable: next city

returns: distance to next destination

Stage 1

s_1	d_1	r_1	$f_1(s_1)$
A, B, C	E*	15	15
A, B, D	E*	13	13
A, B, E	D*	10	10
	C	13	13
A, C, B	E*	13	13
A, C, D	E*	10	10
A, C, E	B	13	13
	D*	11	11
A, D, B	E*	12	12
A, D, C	E*	11	11
A, D, E	B*	10	10
	C	11	11

Stage 2

s_2	d_2	$s_1 = s_2 \otimes d_2$	$r_2 + f_1(s_1)$	$f_2(s_2)$
A, B	C	A,B,C	3+15 =18	
	D	A,B,D	4+13 =17	
	E *	A,B,E	3+10 =13	13
A, C	B	A,C,B	3+13 =16	
	D *	A,C,D	2+10 =12	12
	E	A,C,E	5+11 =16	
A, D	B	A,D,B	4+12 =16	
	C *	A,D,C	2+11 =13	13
	E	A,D,E	4+10 =14	

Stage 3

s_3	d_3	$s_2 = s_3 \otimes d_3$	$r_3 + f_2(s_2)$	$f_3(s_3)$
A	B *	A,B	3 + 13 =16	16
	C *	A,C	4 + 12 =16	16
	D	A,D	6 + 13 =19	

Answer:

A B E D C A
A C D E B A

MULTIPLE OBJECTIVE DECISION MAKING

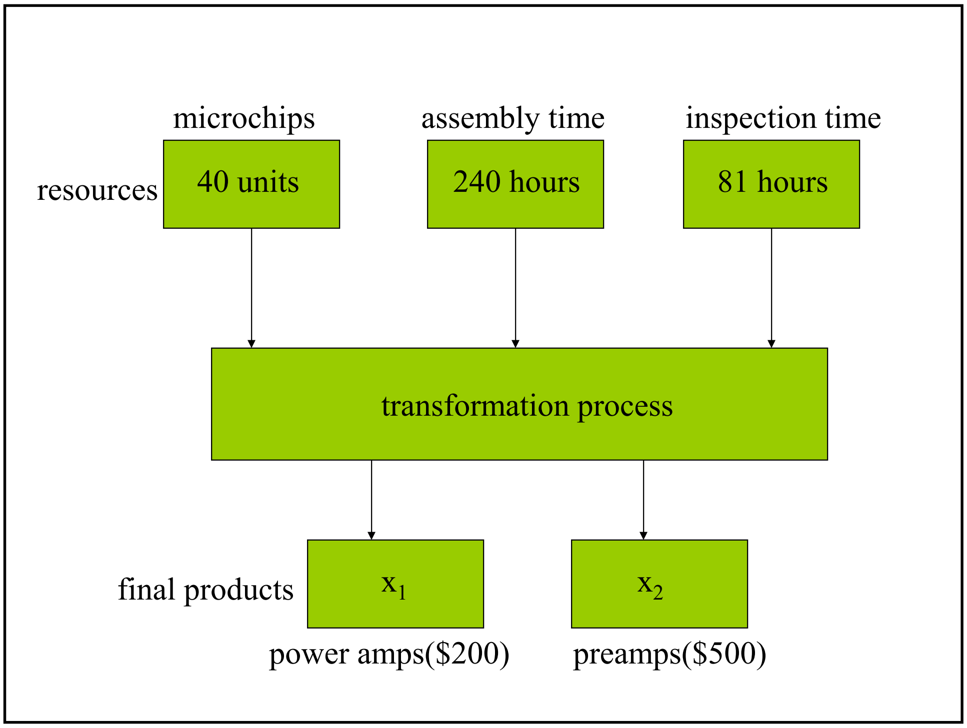
GOAL PROGRAMMING

- Involves satisficing.
- Goals are ranked by order of importance.
- In preemptive goal programming, higher priority goal is assumed to be infinitely more important than a lower priority goal.
- Goal programming achieves as many higher-priority goals as possible, then attempts to get as close as possible to satisfying the remaining goals.

PRODUCT-MIX EXAMPLE

Faze Linear Company is a small manufacturer of high-fidelity components. It has facilities to produce only power amps, only preamps, or a combination of both. Due to limited resources, it is critical to produce appropriate quantities of power amps and/or preamps to maximize profit.

	selling price per unit	profit per unit
power amps	$799.95	$200
preamps	$1000	$500



- Because of a shortage of high quality microchips, at most 40 items can be manufactured on a daily basis.
- All other electronic components are in adequate supply.
- Only 240 hours of assembly worker time is available each day.
- Each power amp requires 1.2 hours of assembly time and preamp requires 4 hours.
- 81 hours are available for testing and inspection each day.
- Power amps require 0.5 hours and preamps require 1 hour of inspection.

- Since power amps do not require a microchip, the limited availability of microchips will directly affect only the number of preamps produced each day. So we have $x_2 \leq 40$.
- Both components require assembly time.
- The combined assembly time must not exceed 240 hours. So we have $1.2x_1+4x_2 \leq 240$.
- For inspection and testing time, the constraint is $0.5x_1+x_2 \leq 81$. Also, there are the nonnegativity conditions $x_1, x_2 \geq 0$. (We ignore integer requirements here.)

LP TO MAXIMIZE PROFIT

Maximize $200x_1 + 500x_2$

s.t.

$$x_2 \leq 40$$

$$1.2x_1 + 4x_2 \leq 240$$

$$0.5x_1 + x_2 \leq 81$$

$$x_1, x_2 \geq 0$$

TWO-GOAL MODEL

- Assume that the Faze Linear management would like to achieve two goals:

  priority 1: achieve profit of $40,000/day.

  priority 2: limit overtime of inspectors.

- For the higher-priority goal, introduce two deviational variables:

  d_1^- = amount by which the target profit of $40,000 is underachieved.

  d_1^+ = amount by which the target profit of $40,000 is overachieved.

- The profit goal as a constraint can now be written as $200x_1+500x_2+ d_1^- - d_1^+ = 40,000$.

- Notice that the $40,000 profit goal need not be met exactly. If d_1^- has a positive value, we will fall short of 40,000 and if d_1^+ has a positive value, we will exceed 40,000.

- By the simplex method, d_1^- and d_1^+ cannot both be positive at the same time.

- Assuming that the 81 hours of inspection time is regular time (not any overtime), we formulate the second goal by defining the deviational variables.
  - d_2^- = amount by which inspection time is underutilized (ie., short of 81 hours)
  - d_2^+ = amount by which inspection time is overutilized (ie., more than 81 hours)
- Formulate the overtime goal as
  $0.5x_1 + x_2 + d_2^- - d_2^+ = 81.$
- Incorporating these goals as constraints in the LP formulation, we obtain the goal programming formulation.

Minimize $P_1 d_1^- + P_2 d_2^+$

s.t.

$$\left.\begin{cases} x_2 \le 40 \text{ microchips} \\ 1.2x_1 + 4x_2 \le 240 \text{ assembly} \end{cases}\right\} \text{resource constraints}$$

$$\left.\begin{cases} 200x_1 + 500x_2 + d_1^- - d_1^+ = 40{,}000 \\ 0.5x_1 + x_2 + d_2^- - d_2^+ = 81 \end{cases}\right\} \text{goal constraints}$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+ \ge 0$$

- The P_1 and P_2 symbols in the objective function reflect the fact that d_1^- and d_2^+ represent priority 1 and 2 goals, respectively.
- Priorities of higher rank are infinitely more important than are those of a lower rank, i.e. $P_1 >> P_2$
- In giving values to P_1 and P_2, take into account the relative size of the units of the variables.

- The deviational variables are to be minimized as they represent the amount by which the goals are not satisfied.
- The result of the ordinal priority rankings is that the profit goal will be achieved to the greatest extent possible before the overtime goal is considered.
- If satisfying any part of the overtime goal causes any reduction in the higher-ranking profit goal, the overtime goal will not be satisfied at all.

HOMEWORK 31

Acme Appliance must determine how many washers and dryers should stocked. It costs Acme \$350 to purchase a washer and \$250 to purchase a dryer. A washer requires 3 sq. yd of storage space, and a dryer requires 3.5 sq. yd. The sale of a washer earns Highland a profit of \$200, and the sale of a dryer \$150. Acme has set the following goals (listed in order of importance):

- Goal 1: Highland should earn at least \$30,000 in profits from the sale of washers and dryers.
- Goal 2: Washers and dryers should not use up more than 400 sq. yd. of storage space.

Formulate a preemptive goal programming model for Acme to determine how many washers and dryers to order. Use D for the number of dryers and W for the number of washers as your variables. There should be only goal constraints in your formulation for this particular problem.

Pareto Optimality

- Example 1. You have two objective functions given as elements in a two-tuple and a decision involving two possible actions. For each action, consider the following returns.

	Return for actions 1 and 2
Action 1	(1, 7)
Action 2	(2, 8)

Obviously, action 2 is better because the return for both objectives is better than those for action 1. Action 1 is said to dominate action 2, so you choose action 1 as your decision.

Pareto Optimality

- Example 2: Again you have two objective functions and a decision involving two possible actions. For each action, consider the following returns.

	Return for actions 1 and 2
Action 1	(8, 2)
Action 2	(9, 1)

This time action 1 is not obviously better than action 2 and vice versa. Neither dominates the other in both returns, so both are said to be Pareto optimal.

Pareto Optimality Optimization Formulation

(P): $vmax \quad (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$

$s.t. \quad \boldsymbol{x} = (x_1, \ldots, x_m)$ ($\boldsymbol{x}$ here is a vector.)

$\boldsymbol{x} \in A \subset R^m$

(S): $max \quad \sum_{i=1}^{n} \alpha_i f_i(\boldsymbol{x})$ (the α_i are scalars)

$s.t. \quad \boldsymbol{x} = (x_1, \ldots, x_m)$

$\boldsymbol{x} \in A \subset R^m$

$\alpha_i > 0, i = 1, \ldots, n$

(P) is the general formulation for a multi-objective optimization problem, whereas (S) is a scalarization of (P) – the parameters α_i reflects the weight attached to objective i. Any solution $\boldsymbol{x}^*$ to (S) is a solution to (P) but not vice versa.

Pareto Optimality – Mathematical Definition

Let $f(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$. Then $\boldsymbol{x}^*$ is a Pareto maximum or an efficient point iff $f(\boldsymbol{x}^*)$ is not dominated by any $\hat{\boldsymbol{x}} \in A$, i.e., there does not exist an $\hat{\boldsymbol{x}} \in A$ for which $f_i(\boldsymbol{x}^*) \leq f_i(\hat{\boldsymbol{x}}), \forall i = 1, \ldots, n$ and there does not exist j such that $f_j(\boldsymbol{x}^*) < f_j(\hat{\boldsymbol{x}})$.

Moreover, let $\alpha_i > 0, i = 1, \ldots, n$. If $\boldsymbol{x}^*$ solves (S), then $\boldsymbol{x}^*$ is an efficient point for (P).

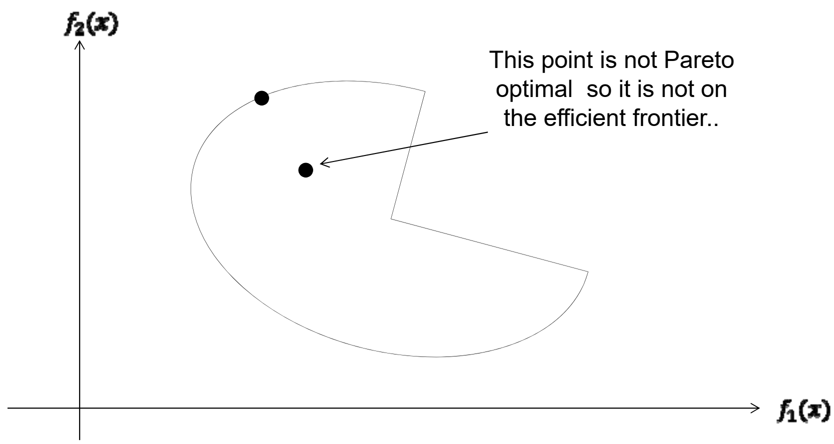
HOMEWORK 32

Consider the Pareto optimization problem

$$\underset{x,y}{\text{Vmax}}\ (2x^2 - 3y^2,\ 25\text{-}5y)$$
$$\text{s.t.}$$
$$0 \leq x \leq 10$$
$$0 \leq y \leq 5,$$

where the first objective function $f_1(x,y) = 2x^2 - 3y^2$ represents profit in hundreds of dollars employee per day and the second objective function $f_2(x,y) = 25\text{-}5y$ represents happiness in the average number of smiles per employee per work day. Is (10,5) an efficient point for this problem? Is (0,0)? Show your work.

Pareto Optimality – Example 1



The efficient frontier of two objective functions can be seen graphically, where the efficient frontier is the set of the objective function values over the feasible region that are not dominated and hence are Pareto optimal two-tuples.

Pareto Optimality – Example 2

Example: Profit Pollution Trade-Off Curve

Problem Statement

Chemco is considering producing three products. The per-unit contribution to profit, labor requirements, raw material used per unit produced, and pollution produced per unit of product are given below. Currently, 1300 labor hours and 1000 units of raw material are available. Chemco's two objectives are to maximize profit and minimize pollution produced. Graph the trade-off curve for this problem.

	Product		
	1	2	3
Profit ($)	10	9	8
Labor (hours)	4	3	2
Raw material (units)	3	2	2
Pollution (units)	10	6	3

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Pareto Optimality – Example 2

Solution

If we define x_i = number of units of product i produced, then Chemco's two objectives may be written as follows:

- Objective 1: Profit $= 10x_1 + 9x_2 + 8x_3$
- Objective 2: Pollution $= 10x_1 + 6x_2 + 3x_3$

We will graph pollution on the x-axis and profit on the y-axis. The values of the decision variables must satisfy the following constraints:

$4x_1 + 3x_2 + 2x_3 \leq 1300$ (Labor constraint) (1)

$3x_1 + 2x_2 + 2x_3 \leq 1000$ (Raw material constraint) (2)

$x_i \geq 0, i = 1,2,3$ (3)

We can find a Pareto optimal solution by choosing to optimize either of our objectives, subject to the above constraints. We begin by maximizing profit. To do this, we must solve the following LP **(LP 1)**:

$$\begin{aligned} \max z = \; & 10x_1 + 9x_2 + 8x_3 \\ s.t. \; & 4x_1 + 3x_2 + 2x_3 \leq 1300 \\ & 3x_1 + 2x_2 + 2x_3 \leq 1000 \\ & x_i \geq 0, i = 1,2,3 \end{aligned}$$

Using TORA or LINGO, the unique optimal solution is $z = 4300$, $x_1 = 0$, $x_2 = 300$ and $x_3 = 200$.

Pareto Optimality – Example 2

Solution (Continued)

Why is the solution ($z = 4300$, $x_1 = 0$, $x_2 = 300$ and $x_3 = 200$) Pareto optimal?

If this solution is not Pareto optimal, there would have to be a solution satisfying (1)-(3) that yielded $z \geq 4300$ and pollution ≤ 2400, with at least one of these inequalities holding strictly. (Please note that this solution yields a pollution level of $6(300) + 3(200) = 2400$ units.) Since it is an unique solution to the above LP, there is no feasible solution besides this satisfying (1)-(3) that can have $z \geq 4300$. Thus this solution cannot be dominated.

To find other Pareto optimal solutions, we choose any level of pollution (POLL) and solve the following LP **(LP 2)**:

$$\begin{aligned} \max z = \; & 10x_1 + 9x_2 + 8x_3 \\ s.t. \; & 4x_1 + 3x_2 + 2x_3 \leq 1300 \\ & 3x_1 + 2x_2 + 2x_3 \leq 1000 \\ & 10x_1 + 6x_2 + 3x_3 \leq POLL \\ & x_i \geq 0, i = 1,2,3 \end{aligned}$$

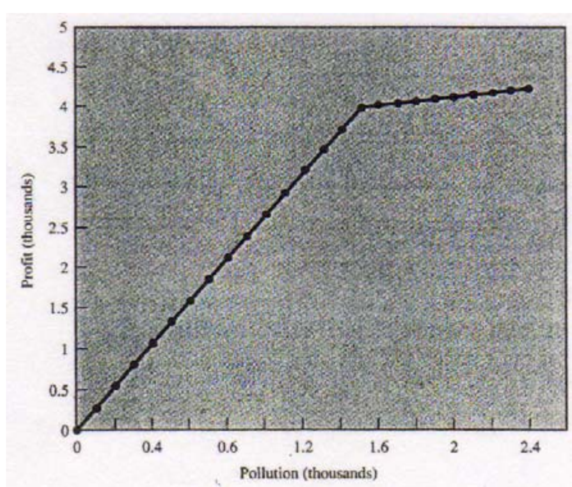
Pareto Optimality – Example 2

Solution (Continued)

Let PROF be the (unique) optimal z-value when this LP is solved. For each value of POLL, the point (POLL, PROF) will be on the trade-off curve. To see this, note that any point (POLL', PROF') dominating (POLL, PROF) must have PROF' >= PROF. The fact that (POLL, PROF) is the unique solution to LP 2 implies that all feasible points (with the exception of [POLL, PROF]) having PROF'>=PROF must have POLL'>POLL.

This means that (POLL, PROF) cannot be dominated, so it is on the trade-off curve. Choosing any value of POLL> 2400 yields no new points on the trade-off curve. Thus, as our next step we choose POLL = 2300. Then using TORA, we can get the optimal z-value of 4266.67 and $10x_1 + 6x_2 + 3x_3 = 2300$. Thus, the point (2300, 4266.67) is on the trade-off curve. Next, we change POLL to 2200 and obtain the point (2200, 4233.33) on the trade-off curve. Continuing in this fashion, setting POLL= 2100, 2000, 1900, ...0, we obtain the trade-off curve between profit and pollution give as follows:

Pareto Optimality – Example 2



REVIEW FOR QUIZ 3